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Solutions of the Schrodinger equation with gravitational plus exponential potential

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ABSTRACT

The parametric Nikiforov-Uvarov method has been used to solve the Schrodinger equation with gravitational plus exponential potential. The energy eigenvalues in bound states and their corresponding un-normalized eigenfunctions are presented.

Keywords

Schrodinger equation, gravitational potential, exponential potential, parametric Nikiforov-Uvarov method.

1 INTRODUCTION

Obtaining bound state solutions of the Schrodinger equation (SE) have been reported to be possible only for certain potentials of physical interest [1-5]. Recently, several authors have attempted to obtain either the exact or approximate solutions of the Schrodinger equation for different potentials [6-10]. It has been shown that a number of these potentials play very important roles in Molecular Physics, Solid State and Chemical Physics [8]. In this work, our aim is to obtain the solution of the Schrodinger equation using the gravitational plus exponential potential (G+EP) of the form:

$$V(z)=mgz+\delta e^{-kz} \qquad (1)$$

Here, the displacement is denoted by z, momentum by *k*, mass by *m*, and gravitational acceleration by *g*. Note that it is possible from the quantum mechanical perspectives, to use the proposed potential, G+EP for the calculation of the energy of a body falling under gravity. Earlier, Berberan-Santos *et al* [11] used the gravitational potential without the exponential term to study the motion of a particle in a gravitational field and obtained the probability distribution function for both classical and quantum mechanical position of the particle. Exponential potentials in the form of the second term of equation (1) have been studied widely in the literature. For example, Andianov *et al* [12] used piecewise exponential potential of the form $V(\varphi)=V_0e^{\lambda}\varphi$ to investigate the general solution of scalar field cosmology. This potential is similar to the second term of equation (1) if we perform the mapping $V_0\rightarrow\delta,\ \lambda\rightarrow -k,\ \varphi\rightarrow z$. Also, Amore and Fernandez [13] obtained accurate complex eigenvalues for the Schrodinger equation using exponential potential similar to the second term of equation (1). However, very little has been achieved in the solution of the Schrodinger equation with the proposed potential, G+EP in the literature.

2 Overview of Nikiforov-Uvarov Method

The The Nikiforov-Uvarov method solves generalized differential equations whose solutions are orthogonal functions [14]. For instance, the Schrodinger equation of the type:

$$\psi(r)+\left[E-V(r)\right]\psi(r)=0 \qquad (2)$$

could be solved by this method. To do so first, equation (2) is transformed with appropriate coordinate transformation, say s = *s*(*r*), into an equation of hypergeometric type :

$$\psi''(s)+\frac{\overline{\tau}(s)}{\sigma(s)}\psi'(s)+\frac{\overline{\sigma}(s)}{\sigma^2(s)}\psi(s)=0 \tag{3}$$

where $\sigma(s)\neq 0$ and $\overline{\sigma}(s)$ are polynomials of atmost second degree, and $\overline{\tau}(s)$ is a first degree polynomial. Equation (3) can then be solved using the parametric NU method. In general, the parametric NU method is given by the generalized hypergeometric-type equation [6]

$$\psi''(s)+\frac{c_1-c_2 s}{s(1-c_3 s)}\psi'(s) + \frac{1}{s^2(1-c_3 s)^2}\left[-\xi_1 s^2+\xi_2 s-\xi_3\right]\psi(s)=0 \tag{4}$$

From equation (4), the eigenfunctions and corresponding eigenvalues of the equation can be obtained as follows

$$\psi(s)=N_n s^{c_{12}}(1-c_3 s)^{-c_{12}-\frac{c_{13}}{c_3}} P_n^{\left(c_{10}-1,\frac{c_{11}}{c_3}-c_{10}-1\right)} \tag{5}$$

$$(c_2-c_3)n+c_3 n^2-(2n+1)c_5 +(2n+1)\left(\sqrt{c_9}+c_3\sqrt{c_8}\right) +c_7+2c_3 c_8+2\sqrt{c_8 c_9}=0 \tag{6}$$

where

$$\begin{aligned} &c_4=\frac{1}{2}(1-c_1), c_5=\frac{1}{2}(c_2-2c_3),\\ &c_6=c_5^2+\xi_1, c_7=2c_4c_5-\xi_2,\\ &c_8=c_4^2+\xi_3 c_9=c_3c_7+c_2^2c_8+c_6,\\ &c_{10}=c_1+2c_4+2\sqrt{c_8},\\ &c_{11}=c_2-2c_5+2\left(\sqrt{c_9}+c_3\sqrt{c_8}\right),\\ &c_{12}=c_4+\sqrt{c_8},\\ &c_{13}=c_5-\left(\sqrt{c_9}+c_3\sqrt{c_8}\right), \end{aligned} \tag{7}$$

N_n is the normalization constant and $P_n^{(\alpha,\beta)}$ are the Jacobi polynomials.

3 The Schrodinger Equation

The The Schrodinger equation with the potential $V(r)$, is given in spherical coordinates (see [6]) as

$$-\frac{\hbar^2}{2\pi}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) +\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\theta^2}\right]\psi(r,\theta,\phi) +V(r)\psi(r,\theta,\phi)=E\psi(r,\theta,\phi) \tag{8}$$

Using the common ansatz for the wave function:

$$\psi(r,\theta,\phi)=\frac{R(r)}{r}Y_{lm}(\theta,\phi) \tag{9}$$

in equation (8) yields the following set of equations:

$$\frac{d^2R_{nl}(r)}{dr^2}+\frac{2\mu}{\hbar^2}\left[E-V(r)-\frac{\lambda\hbar^2}{2\mu r^2}\right]R_{nl}(r)=0 \tag{10}$$

$$\frac{d^2\Theta_{ml}(\theta)}{d\theta^2}+\cot\theta\frac{d\Theta_{ml}(\theta)}{d\theta}\left(\lambda-\frac{m^2}{\sin^2\theta}\right)\Theta_{ml}(\theta) =0 \tag{11}$$

$$\frac{d^2\Phi_m(\phi)}{d\phi^2}+m^2\Phi_m(\phi)=0 \tag{12}$$

where $\lambda=l(l+1)$ and m^2 are the separation constants. $Y_{lm}(\theta,\phi)=\Theta_{ml}(\theta)\Phi_m(\phi)$ is the solution of equations (11) and (12) which are the well known spherical harmonic functions [15].

4 Solutions to the Radial equation

In what follows, we proceed to solve the radial part of the Schrodinger equation that is, equation (10). The potential in equation (1) can be written as

$$V(r)=\beta r+V_0 e^{-\alpha r}, \tag{13}$$

where $\beta=mg, \alpha=k, z=r, \delta=V_0$. We can also write equation (13) as

$$V(r)=\beta r+V_0\left(1-\alpha r+\alpha^2r^2\right) \qquad (14)$$

On rearranging equation (14) we get our working potential amenable to NU method as

$$V(r)=V_0+(\beta-\alpha V_0)r+\alpha^2V_0r^2 \qquad (15)$$

To see the accuracy of our approximation, the G+EP (13) and its approximation (15) are plotted as functions of *r* using the parameters *β* = 1, and *V0* = 0.5*f m*−1 for different momenta namely; *α* = 0.2, 0.4 and 0.6 respectively in Figure 1.

Various quantum mechanical equations such as the Schrodinger equation (SE), Klein-Gordon equation (KG) and Dirac equation can be solved using this new potential (15) by the NU method for their exact solutions. Writing equation (10) as

$$\frac{d^2R(r)}{dr^2}+\frac{2\mu}{\hbar^2}\left[E-V(r)-\frac{l(l+1)}{2\mu r^2}\right]R(r)=0 \qquad (16)$$

and using equation (15) in equation (16) yields

$$\frac{d^2R(r)}{dr^2}+\frac{2\mu}{\hbar^2}\left[\begin{array}{l}E-V_0-(\beta-\alpha V_0)r\\ -\alpha^2V_0r^2-\dfrac{l(l+1)}{2\mu r^2}\end{array}\right]R(r)=0 \qquad (17)$$

Equation (17) can only be solved exactly for s-wave, that is, $l=0$ and we get

$$\frac{d^2R(r)}{dr^2}+\frac{2\mu}{\hbar^2}\left[\begin{array}{l}E-V_0-(\beta-\alpha V_0)r\\ -\alpha^2V_0r^2\end{array}\right]R(r)=0 \qquad (18)$$

Equation (18) can be rewritten as

$$\frac{d^2R(r)}{dr^2}+\left[\frac{2\mu}{\hbar^2}(E-V_0)-\frac{2\mu}{\hbar^2}(\beta-\alpha V_0)r -\frac{2\mu}{\hbar^2}\alpha^2V_0r^2\right]R(r)=0 \qquad (19)$$

Comparing equation (19) with equation (4) yields the following parameters

$$\begin{aligned}&c_1=c_2=c_3=0,\\ &\xi_1=\frac{2\mu\alpha^2V_0}{\hbar^2},\\ &\xi_2=\frac{2\mu(\beta-\alpha V_0)}{\hbar^2},\\ &\xi_3=-\frac{2\mu(E-V_0)}{\hbar^2}\end{aligned} \qquad (20)$$

Other coefficients are determined as

$$\begin{aligned}&c_4=\frac{1}{2},\ c_5=0,\ c_6=\xi_1,\ c_7=-\xi_2,\\ &c_8=\frac{1}{4}+\xi_3,\ c_9=\xi_1,\ c_{10}=1+2\sqrt{\tfrac{1}{4}+\xi_3},\\ &c_{11}=2\sqrt{\xi_1},c_{12}=\frac{1}{2}+\sqrt{\tfrac{1}{4}+\xi_3},\ c_{13}=-\sqrt{\xi_1}\end{aligned} \qquad (21)$$

Now using equations (6), (20) and (21) we obtain the energy spectrum of the G+EP as

$$E_n=-\frac{\hbar^4}{4\mu^2\alpha^2V_0}\left\{\left(n+\frac{1}{2}\right)\left(\sqrt{\frac{2\mu\alpha^2V_0}{\hbar^2}}\right) +\left[\frac{\mu(\beta-\alpha V_0)}{\hbar^2}\right]^2-\frac{\mu\alpha^2V_0}{2\hbar^2}\right\}+V_0 \qquad (22)$$

Numerical experiments conducted with the energy spectrum of the gravitational plus exponential potential (22) yielded the results shown in the Table 1. In order to carry out the experiment, we used the following values for the parameters namely; $V_0=0.5\,fm^{-1}$ and $\hbar=\beta=\mu=1$.

The radial wave function is then obtained from equation (5) as

$$R_{nl}(s)=N_ns^{\frac{1}{2}+\frac{\varepsilon}{2}}L_n^{\varepsilon}\left(2\sqrt{\xi_1}s\right) \qquad (23)$$

where $\varepsilon=2\sqrt{\frac{1}{4}+\xi_3}$ and $L_n^{\varepsilon}\left(2\sqrt{\xi_1}s\right)$ is the Laguerre polynomial.

5 Conclusion

The energy eigenvalues and their corresponding un-normalized wavefunctions have been obtained using the parametric NU method for the Schrodinger equation with the gravitational plus exponential potential.

Table 1: The energy Eigenvalues (in fm^{-1}) of the G+EP potential.

n	Energy Spectrum for α = 0.2	Energy Spectrum for α = 0.4	Energy Spectrum for α = 0.6
0	-10.750000000	-2.000000000	-0.472222222
1	-13.250000000	-3.250000000	-1.305555556
2	-15.750000000	-4.500000000	-2.138888889
3	-18.250000000	-5.750000000	-2.972222222
4	-20.750000000	-7.000000000	-3.805555556
5	-23.250000000	-8.250000000	-4.638888889
6	-25.750000000	-9.500000000	-5.472222222
7	-28.250000000	-10.75000000	-6.305555556
8	-30.750000000	-12.00000000	-7.138888889
9	-33.250000000	-13.25000000	-7.972222222
10	-35.750000000	-14.50000000	-8.805555556

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