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Solutions to the Klein-Gordon Equation with Inversely Quadratic Yukawa plus Inversely Quadratic Potential using Nikiforov-Uvarov Method

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ABSTRACT

The solutions of the Klein-Gordon equation with inversely quadratic Yukawa plus inversely quadratic potential have been obtained using the parametric Nikiforov-Uvarov (NU) method which is based on the solutions of general second-order linear differential equations with special functions. The bound state energy eigenvalues and the corresponding un-normalized eigen functions are obtained in terms of Jacobi polynomials. Also special cases of the potential have been considered and their energy eigen values obtained.

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Keywords

Klein-Gordon equation, inversely quadratic Yukawa potential, inversely quadratic potential, Nikiforov-Uvarov method, Jacobi polynomials

1 INTRODUCTION

There has been steady interest in finding the analytic solutions to the Klein-Gordon (KG) equation using various methods and potentials [1 – 10]. These analytic solutions play an important role in quantum mechanics since they contain all the necessary information concerning the quantum system under consideration [11]. Furthermore, the exact solutions of the KG equation with some vector and scalar potentials is of interest due to the fact that any strong potential field determines the relativistic effects and gives correction to the non-relativistic results [11]. The inversely quadratic Yukawa potential was first studied in 2012 by Hamzavi et al [12] when they obtained approximate spin and pseudospin solutions to the Dirac equation with the potential including a tensor interaction. Since then several papers on the potential have appeared in the literature [13 – 15]. The inversely quadratic potential has been used by Oyewumi and Bangudu [16] in combination with isotropic harmonic oscillator in N-dimension spaces. However, not much has been achieved in the area of solving the relativistic Klein-Gordon equation with inversely quadratic Yukawa plus inversely quadratic potential (IQYIQP) using Nikiforov-Uvarov (NU) method which is the purpose of this paper.

The paper is organized as follows: Section 1 has the introduction, the NU method is reviewed in section 2. In section 3, the Klein-Gordon equation with IQYIQP is solved using the NU method. Finally, we give a brief discussion in section 4 before the conclusion in section 5 and then the references also given.

2 NIKIFOROV-UVAROV METHOD

The Nikiforov-Uvarov (NU) method is based on the solutions of a generalized second-order linear differential equation with special orthogonal functions [17]. It is applied to Schrödinger equation and Schrödinger-like equations of the type as:

$$\psi''(r) + [E - V(r)]\psi(r) = 0, \tag{1}$$

The solutions to equation (1) can be obtained by transforming it into an equation of hypergeometric type with appropriate coordinate transformation $s = s(r)$ to get

$$\psi''(s)+\frac{\bar{\tau}(s)}{\sigma(s)}\psi'(s)+\frac{\bar{\sigma}(s)}{\sigma^2(s)}\psi(s)=0, \tag{2}$$

To solve equation (2) we can use the parametric NU method. The parametric generalization of the NU method is expressed by the generalized hypergeometric type equation [18]

$$\psi''(s)+\frac{(c_1-c_2 s)}{s(1-c_3 s)}\psi'(s)+\frac{1}{s^2(1-c_3 s)^2}[-\epsilon_1 s^2+\epsilon_2 s-\epsilon_3]\psi(s)=0, \tag{3}$$

where $\sigma(s)$ and $\bar{\sigma}(s)$ are polynomials atmost second degree, and $\bar{\tau}(s)$ is a first degree polynomial. The eigen functions (equation 4) and corresponding eigenvalues (equation 5) to the equation (3) become

$$\psi(s)=N_n s^{c_{12}}(1-c_3 s)^{-c_{12}-\frac{c_{13}}{c_3}}P_n^{\left(c_{10}-1,\frac{c_{11}}{c_3}-c_{10}-1\right)}(1-2c_3 s), \tag{4}$$

$$(c_2-c_3)n+c_3 n^2-(2n+1)c_5+(2n+1)\left(\sqrt{c_9}+c_3\sqrt{c_8}\right)+c_7+2c_3 c_8+2\sqrt{c_8 c_9}=0, \tag{5}$$

Where

$$c_4=\frac{1}{2}(1-c_1), c_5=\frac{1}{2}(c_2-2c_3), c_6={c_5}^2+\epsilon_1, c_7=2c_4 c_5-\epsilon_2, c_8={c_4}^2+\epsilon_3, \quad c_9=c_3 c_7+{c_2}^2 c_8+c_6, c_{10}=c_1+2c_4+2\sqrt{c_8}, c_{11}=c_2-2c_5+2\left(\sqrt{c_9}+c_3\sqrt{c_8}\right), c_{12}=c_4+\sqrt{c_8},$$

$$c_{13}=c_5-\left(\sqrt{c_9}+c_3\sqrt{c_8}\right), \tag{6}$$

N_n is the normalization constant and $P_n^{(\alpha,\beta)}$ are the Jacobi polynomials.

3 SOLUTIONS OF THE KLEIN-GORDON EQUATION

The Klein-Gordon equation with equal scalar potential $S(r)$ and vector potential $V(r)$ in natural units ($\hbar=c=1$) is given as [19]

$$\frac{d^2R(r)}{dr^2}+[(E^2-M^2)-2(E+M)V(r)]R(r)=0, \tag{7}$$

Where M is the rest mass and E is the relativistic energy.

The inversely quadratic Yukawa potential (IQYP) is given as [12]

$$V(r)=-\frac{V_0 e^{-2\alpha r}}{r^2}, \tag{8}$$

Where V_0 is the depth of the potential and α is a real parameter. Similarly, the inversely quadratic potential (IQP) is given as [16]

$$V(r)=-\frac{V_0'}{r^2}, \tag{9}$$

Making the transformation $s=e^{-2\alpha r}$ the sum of the potentials in equations (8) and (9) becomes

$$V(s)=-\frac{V_0 s4\alpha^2}{(1-s)^2}-\frac{V_0' 4\alpha^2}{(1-s)^2} \tag{10}$$

We have used the approximation $\frac{1}{r^2}\approx\frac{4\alpha^2}{(1-s)^2}$ in equation (10) to enable us solve equation (7).

Again, applying the transformation $s=e^{-2\alpha r}$ to get the form that NU method is applicable, equation (7) gives a generalized hypergeometric-type equation as

$$\frac{d^2R(s)}{ds^2}+\frac{(1-s)}{(1-s)s}\frac{dR(s)}{ds}+\frac{1}{(1-s)^2 s^2}[-\beta^2 s^2+(2\beta^2+A)s-(\beta^2+B)]R(s)=0, \tag{11}$$

Where

$$-\beta^2=\left(\frac{E^2-M^2}{4\alpha^2}\right), A=2(E+M)V_0, B=2(E+M){V_0}', \tag{12}$$

Comparing equation (11) with equation (3) yields the following parameters

$$c_1=c_2=c_3=1, c_4=0, c_5=-\frac{1}{2}, c_6=\frac{1}{4}+\beta^2, c_7=-2\beta^2-A, c_8=\beta^2-B, c_9=\frac{1}{4}-A-B, c_{10}=1+2\sqrt{\beta^2-B}, c_{11}=2+2\left(\sqrt{\frac{1}{4}-A-B}+\sqrt{\beta^2-B}\right), c_{12}=\sqrt{\beta^2-B}, c_{13}=-\frac{1}{2}-\left(\sqrt{\frac{1}{4}-A-B}+\sqrt{\beta^2-B}\right), \epsilon_1=\beta^2, \epsilon_2=2\beta^2+A, \epsilon_3=\beta^2-B, \tag{13}$$

Now using equations (5), (12) and (13) we obtain the energy eigen spectrum of the IQYIQP as

$$\beta^2=\left[\frac{(2B+A)-\left(n^2+n+\frac{1}{2}\right)-(2n+1)\sqrt{\frac{1}{4}-A-B}}{(2n+1)+2\sqrt{\frac{1}{4}-A-B}}\right]^2+B, \tag{14}$$

With the substitution of values, equation (14) yields the explicit energy eigen spectrum of IQYIQP as

$$E^2-M^2=-4\alpha^2\left\{\left[\frac{4(E+M)V_0'+2(E+M)V_0-\left(n^2+n+\frac{1}{2}\right)-(2n+1)\sqrt{\frac{1}{4}-2(E+M)V_0-2(E+M)V_0'}}{(2n+1)+2\sqrt{\frac{1}{4}-2(E+M)V_0-2(E+M)V_0'}}\right]^2\right\}-8\alpha^2(E+M)V_0', \tag{15}$$

We now calculate the radial wave function of the IQYIQP as follows

The weight function $\rho(s)$ is given as [19]

$$\rho(s) = s^{c_{10}-1}(1-c_3 s)^{\frac{c_{11}}{c_3}-c_{10}-1}, \quad (16)$$

Using equation (13) we get the weight function as

$$\rho(s) = s^u(1-s)^v, \quad (17)$$

Where $u = 2\sqrt{\beta^2 - B}$ and $v = 2\sqrt{\frac{1}{4} - A - B}$

Also we obtain the wave function $\chi_n(s)$ as [19]

$$\chi_n(s) = P_n^{c_{10}-1,\frac{c_{11}}{c_3}-c_{10}-1}(1-2c_3 s), \quad (18)$$

Using equation (13) we get the function χ(s) as

$$\chi_n(s) = P_n^{(u,v)}(1-2s), \quad (19)$$

Where $P_n^{(u,v)}$ are Jacobi polynomials

Lastly,

$$\varphi(s) = s^{c_{12}}(1-c_3 s)^{-c_{12}-\frac{c_{13}}{c_3}}, \quad (20)$$

And using equation (20) we get

$$\varphi(s) = s^{u/2}(1-s)^{1+v/2}, \quad (21)$$

We then obtain the radial wave function from the equation [19]

$$R_n(s) = N_n \varphi(s)\chi_n(s), \quad (22)$$

As

$$R_n(s) = N_n s^{u/2}(1-s)^{1+v/2} P_n^{(u,v)}(1-2s), \quad (23)$$

Where n is a positive integer and N_n is the normalization constant.

4 DISCUSSION

We consider the following cases from equation (15)

Case 1: If we choose $V_0' = 0$ then the energy eigen values of the IQY potential become

$$E^2 - M^2 = -4\alpha^2\left\{\left[\frac{2(E+M)V_0 - \left(n^2+n+\frac{1}{2}\right) - (2n+1)\sqrt{\frac{1}{4}-2(E+M)V_0}}{(2n+1)+2\sqrt{\frac{1}{4}-2(E+M)V_0}}\right]^2\right\} \quad (24)$$

Case 2: If we choose $V_0 = 0$ then the energy eigen values of the IQ potential become

$$E^2 - M^2 = -4\alpha^2\left\{\left[\frac{4(E+M)V_0' - \left(n^2+n+\frac{1}{2}\right) - (2n+1)\sqrt{\frac{1}{4}-2(E+M)V_0'}}{(2n+1)+2\sqrt{\frac{1}{4}-2(E+M)V_0'}}\right]^2\right\} - 8\alpha^2(E+M)V_0' \quad (25)$$

5 CONCLUSION

The Klein-Gordon (KG) equation for the sum of IQYP and IQP referred to as IQYIQP has been solved and solutions obtained using the Nikiforov-Uvarov (NU) method. Special cases of the potential have also been considered and their eigen values obtained.

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