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Bound states of the Klein- Gordon equation for spin symmetry with exponential-type potentials by the Supersymmetry method

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ABSTRACT

In this work, the Klein-Gordon equation is studied with exponential-type potentials to $l \neq 0$ states. As regards, the analytical solutions of the Klein-Gordon equation are possible only in the s-wave case with the angular momentum $l = 0$ for some well known potential. We have applied the Supersymmetry method (SUSY) by using the Pekeris approximation and we are obtained bound state spectrum and wave function of exponential-type potential for nonzero angular momentum. Spin symmetry and the shape invariance method are used in the calculation.

Keywords

Klein-Gordon equation, exponential-type potentials, Supersymmetry method (SUSY).

1. INTRODUCTION

It is well known that analytical solution of the Klein-Gordon equation and Dirac equation play an important role in relativistic quantum mechanics, because the wave function contains all information to description of a quantum system [1-6]. The Klein-Gordon equation describe the motion of a spin- zero particle [7-9]. The energy levels and eigenvalues of Klein-Gordon equation have determinate with simple potentials [10-12].

However, the analytical solutions of the Klein-Gordon equation are possible only in the s-wave case with the angular momentum $l = 0$ for some well known potential [١٣-14]. Conversely, when $l \neq 0$, one can only solve approximately the Klein-Gordon equation for some potentials using a suitable approximation scheme [15-16].

These exponential-type potentials such as Titz- Wei potential are widely used in many branches of physics. A typical example is the Hulthén potential, which is used in nuclear physics, atomic physics, solid state physics, and chemical physics. The Titz-Wei potential of the form is [17-18]

$$V(r) = V_0 \left(\frac{1-e^{-2\alpha r}}{1-qe^{-2\alpha r}}\right)^2 \qquad (1)$$

Where V_0 is potential depth, *q* variation parameter and the range of 0 < *q* < 1. We can not exact solution of Klein-Gordon equation so we have applied the Supersymmetry method (SUSY) by using the Pekeris approximation to obtain bound state spectrum and wave function of the potential in Eq. (1) for nonzero angular momentum.

In the Klein-Gordon equation and Dirac equation, pseudospin symmetry and spin symmetry occur for Σ(*r*) = V(r) + S(r) = const and Δ(*r*) = V(r) – S(r) = const respectively which V(r) is vector potential and S(r) is scalar potential. Spin symmetry and pseudospin symmetry have been found interesting applications in the field of nuclear physics [19-24].

The purpose of this work is to solve approximately the arbitrary *l*-state Klein-Gordon equation with SUSY method and spin symmetry in exponential-type potential. We have been obtained the energy eigenvalues equation and the corresponding spinor wave functions by using the six-parameter exponential- type potentials (SPEP) [25].

2. Three dimension Klein-Gordon equation

The three dimensional Klein-Gordon equation with mixed vector V(r) and scalar S(r) potentials can be written as

$$\left[\nabla^2 + (V(r)-E)^2 - (S(r)+M)^2\right]\psi(r,\theta,\varphi) = 0 \qquad (2)$$

Where *M* is the rest mass, *E* is the relativistic energy. In spherical coordinates, the Klein-Gordon equation becomes

$$[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r})+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta})+$$
$$\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}-2\{EV(r)+MS(r)\}+V^2(r)-S^2(r)$$
$$+E^2-M^2]\psi(r,\theta,\varphi)=0 \quad (3)$$

If one assigns the corresponding spherical total wave function as

$$\psi(r,\theta,\varphi)=\frac{R(r)}{r}Y_{lm}(\theta,\varphi) \quad (4)$$

Where

$$Y_{lm}(\theta,\varphi)=\Theta(\theta)\Phi(\varphi) \quad (5)$$

Then the wave equation in Eq. (3) is separated into variables and the following equations are obtained:

$$\frac{d^2R(r)}{dr^2}+[E^2-M^2-2\{EV(r)+MS(r)\}$$
$$+V^2(r)-S^2(r)-\frac{\lambda}{r^2}]R(r)=0 \quad (6)$$

$$\frac{d^2\Theta(\theta)}{d\theta^2}+Cot\theta\frac{d\Theta(\theta)}{d\theta}(\lambda-\frac{m^2}{\sin^2\theta})\Theta(\theta)=0 \quad (7)$$

$$\frac{d^2\Phi(\varphi)}{d^2\varphi}+m^2\Phi(\varphi)=0 \quad (8)$$

Where m^2 and $\lambda=l(l+1)$ are the separation constants. The radial part of Klein-Gordon equation in Eq. (6) for spin symmetry and substituting Eq. (1) into Eq. (6) is written as

$$\frac{d^2R(r)}{dr^2}+[E^2-M^2-2\{E+M\}V_0(\frac{1-e^{-2\alpha r}}{1-qe^{-2\alpha r}})^2$$
$$-\frac{\lambda}{r^2}]R(r)=0 \quad (9)$$

this equation cannot be solved analytically for $l\neq 0$ due to the centrifugal term because the equation is a combination of the exponential and inverse square potentials. Therefore, Eq. (9) can be evaluated by using an improved approximation scheme so we used Pekeris approximation. This approximation is based on the expansion of the spin-orbit coupling term in a series of exponentials depending on the intern clear distance [26-28] so Eq. (9) takes the form of Schrödinger equation

$$\frac{1}{r^2}\cong 4\alpha^2\left[C_0+\frac{e^{-2\alpha r}}{\left(1-qe^{-2\alpha r}\right)^2}\right] \quad (10)$$

Where C is a dimensionless constant. Substituting Eq. (10) into Eq. (9), we obtain

$$\frac{d^2R(r)}{dr^2}+[E^2-M^2-2\{E+M\}V_0(\frac{1-e^{-2\alpha r}}{1-qe^{-2\alpha r}})^2$$
$$-4\alpha^2\lambda\{C_0+\frac{e^{-2\alpha r}}{\left(1-qe^{-2\alpha r}\right)^2}\}]R(r)=0 \quad (11)$$

We can write the Schrödinger-like Eq. (11)

$$[-\frac{d^2}{dr^2}+\tilde{V}_1\frac{e^{-2\alpha r}}{(1-qe^{-2\alpha r})^2}+\tilde{V}_2(\frac{1-e^{-2\alpha r}}{1-qe^{-2\alpha r}})^2]R(r)$$
$$=\tilde{E}_0R(r) \quad (12)$$

Where

$$\tilde{V}_1=4\alpha^2\lambda$$
$$\tilde{V}_2=2(E+M)V_0 \quad (13)$$
$$\tilde{E}_0=E^2-M^2-4\alpha^2\lambda C_0$$

In section 3 we would introduce the Supersymmetry method to solve Eq. (12)

3. Supersymmetry method (SUSY)

In Supersymmetry Quantum Mechanics (SUSYQM) we normally deal with the partner Hamiltonians

$$H_\pm=\frac{P^2}{2m}+V_\pm(r) \quad (14)$$

Where

$$V_\pm(r)=W^2(r)\pm\frac{d}{dr}W(r) \quad (15)$$

Where $W(r)$ is called a superpotential in SUSY method. We write down the ground-state lower spinor component $R_{0,k}(r)$ as

$$R_{0,k}(r)=\exp\left(-\int W(r)dr\right) \quad (16)$$

The supersymmetric partner potentials $V_+(r)$ and $V_-(r)$ are given by

$$V_+(r)=W^2(r)+\frac{dW}{dr} \quad (17)$$

$$V_-(r)=W^2(r)-\frac{dW}{dr} \quad (18)$$

If $V_+(r)$ and $V_-(r)$ have similar shapes, they are said to be shape-invariant. And they satisfy the following relation [29]

$$V_+(r,a_0)=V_-(r,a_1)+R(a_1) \quad (19)$$

Where a_0 is a set of parameters, a_1 is a function of a_0, the remainder $R(a_1)$ is independent of *r* and $E_0^{(-)}=0$, we obtain

$$E_n^{(-)}=\sum_{k=1}^{n}R(a_k)=R(a_1)+R(a_2)+\ldots+R(a_n)$$
$$E_n=E_n^{(-)}+E_0 \quad (20)$$

4. Energy levels of Klein-Gordon equation with Titz-Wei potential

The energy levels are obtained by using the six-parameter exponential- type potential (SPEP) and the supersymmetric shape invariance technique. Substituting Eq. (16) into Eq. (12), we obtain

$$W^2(r)-\frac{dW(r)}{dr}=\bar{V}_1\left(\frac{e^{-2\alpha r}}{\left(1-qe^{-2\alpha r}\right)^2}\right)$$
$$+\bar{V}_2\left(\frac{1-e^{-2\alpha r}}{1-qe^{-2\alpha r}}\right)^2-\bar{E}_0 \quad (21)$$

Eq. (21) is a non-linear Riccati equation so by using the SUSY method we can put the superpotential as

$$W(r)=W(x-x_e)=-\left(Q_1+\frac{Q_2}{e^{b(x-x_e)}-qe^{bx_e}}\right) \quad (22)$$

Where in Eq. (2٢) we put $r=x-x_e$ which x_e is the dimensionless constant and

$$Q_1=\frac{1}{2qQ_2}\{q(2\bar{V}_2(q-1)e^{bx_e}+\bar{V}_1e^{bx_e})$$
$$+(\bar{V}_2(q-1)^2e^{2bx_e}+\bar{V}_1e^{2bx_e})-Q_2^2\}$$
$$Q_2=q\{-\frac{b}{2}\pm[\frac{b^2}{4}+\frac{1}{q^2}(\bar{V}_2(q-1)^2e^{2bx_e}+\bar{V}_1e^{2bx_e})]^{\frac{1}{2}}\}$$
$$(23)$$

Now as for have $W(r)$ and by using Eq. (17) and Eq. (18) we can obtain $V_+(r)$ and $V_-(r)$ as below

$$V_+(x-x_e)=(Q_1+\frac{Q_2}{e^{b(x-x_e)}-qe^{bx_e}})^2+\frac{bQ_2e^{b(x-x_e)}}{(e^{b(x-x_e)}-qe^{bx_e})^2}$$
$$V_-(x-x_e)=(Q_1+\frac{Q_2}{e^{b(x-x_e)}-qe^{bx_e}})^2-\frac{bQ_2e^{b(x-x_e)}}{(e^{b(x-x_e)}-qe^{bx_e})^2}$$
$$(24)$$

Then we can denote $R(a_1)$ by setting $V_+(r)$ and $V_-(r)$ in Eq. (19). Finally by using Eq. (20) energy levels have been obtained as below

$$\bar{E}_n=\bar{V}_2-\frac{(1-q^2)b^2}{16q}$$
$$\times\left[\frac{\bar{V}_2}{-\frac{b^2}{4}(2n+1+\frac{1}{q}\sqrt{q^2+\frac{4(q-1)}{b^2}\bar{V}_2+\frac{4q}{b^2}\bar{V}_1})}\right.$$
$$\left.+(2n+1+\frac{1}{q}\sqrt{q^2+\frac{4(q-1)}{b^2}\bar{V}_2+\frac{4q}{b^2}\bar{V}_1})\right]^2 \quad n=0,1,2,3,\ldots$$
$$(25)$$

Substituting Eq. (13) into Eq. (2٥), we obtain

$$E^2-M^2-4\alpha^2\lambda C_0=2(E+M)V_0-\frac{(1-q^2)b^2}{16q}$$
$$\times\left[\frac{2(E+M)V_0}{-\frac{b^2}{4}(2n+1+\frac{1}{q}\sqrt{q^2+\frac{8(q-1)}{b^2}(E+M)V_0+\frac{16q}{b^2}\alpha^2\lambda})}\right.$$
$$\left.+(2n+1+\frac{1}{q}\sqrt{q^2+\frac{8(q-1)}{b^2}(E+M)V_0+\frac{16q}{b^2}\alpha^2\lambda})\right]^2$$
$$n=0,1,2,3,\ldots \quad (26)$$

If $q=0.1$, $C_0=\frac{1}{12}$ and $l=1$ in Eq. (2٦), so

$$E^2-M^2-\frac{2}{3}\alpha^2=2(E+M)V_0-2.475\alpha^2$$
$$\times\left[\frac{2(E+M)V_0}{-\alpha^2(2n+1+10\sqrt{0.81-\frac{1.98}{\alpha^2}(E+M)V_0}}\right.$$
$$\left.+(2n+1+10\sqrt{0.81-\frac{1.98}{\alpha^2}(E+M)V_0}\right]^2 \quad n=0,1,2,3,\ldots$$

(27)

With solution the radial wave function Eq. (12) we obtain

$$R(r)=N(e^{-2\alpha r})^{\frac{\mu}{2}}(1-qe^{-2\alpha r})^{\frac{1+\nu}{2}}P_n^{(\mu+\nu)}\times(1-2qe^{-2\alpha r})$$
$$(28)$$

Where in Eq. (٢٨), μ and ν are defining as below

$$\mu=2i\sqrt{\frac{1}{4\alpha^2}(E^2-M^2)-l(l+1)C_0} \quad (29)$$
$$\nu=\sqrt{\frac{1}{4}-l(l+1)(C_0+q^2)+\frac{1}{2\alpha^2q}(E+M)} \quad (30)$$

Where N is the normalization constant and P_n is legendre functions. Hence, the total wave function $\psi(r,\theta,\varphi)$ for the potential in Eq. (1) is obtained using Eq. (4) as

$$\psi(r,\theta,\varphi)=N\frac{1}{r}(e^{-2\alpha r})^{\frac{\mu}{2}}(1-qe^{-2\alpha r})^{\frac{1+\nu}{2}}P_n^{(\mu+\nu)}$$
$$\times(1-2qe^{-2\alpha r})Y_{lm}(\theta,\varphi) \quad (31)$$

Finally, as regards the wave function contains all information to description of a quantum mechanics system so it is very important to have obtained.

5. Conclusion

In this paper, we have solved the three dimensional Klein-Gordon equation with the exponential-type potential by applying SUSY method. Using the Pekeris approximation to none centrifugal potential $l\neq 0$ states, we have obtained the analytical eigenfunction and eigenvalue for the exponential-type potential in the Klein-Gordon equation. We can deduce that our results are interesting not only for theoretical physicist also for experimental physicist, because are more general and useful to study nuclear charge radius, spin, nuclear scattering and etc.

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