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Dimension theoretical study on skew product maps with coupled-expanding property

Jinhyon Kim, Hyonhui Ju

Department of Mathematics, **Kim Il Sung** University, Pyongyang, D.P.R.Korea

ABSTRACT

We discuss on some families of skew product maps on a square. For a kind of skew product maps with coupled-expanding property, we estimate Hausdorff dimension of its attractor. And we prove that there exists an ergodic measure with full Hausdorff dimension for this kind of skew product maps.

Keywords

skew product map, Hausdorff dimension, coupled-expanding property, chaotic dynamical system

1. INTRODUCTION

In this paper we study dimensional theoretical properties of a kind of skew product maps with coupled-expanding property. By dimensional theoretical properties we mean Hausdorff dimension of invariant sets of dynamical systems and ergodic measures on these sets.

In the modern theory of dynamical systems geometrical invariants like Hausdorff dimension of invariant sets and measures seem to have their place beside classical invariants like entropy and Lyapnov exponents [4]. In the dimension theoretical study on dynamical systems, following two problems are regarded as important: The first one is to estimate Hausdorff dimension of invariant sets. The second one is the problem about existence of ergodic measure with full Hausdorff dimension in these invariant sets. If such an ergodic measure does not exist, it is problem whether at least the variational principle for Hausdorff dimension holds or not. The variational principle means that there exists a sequence of ergodic measures such that the Hausdorff dimension of the invariant set can be approximated by their Hausdorff dimensions. It seems to be well accepted by experts that these questions are of great important in developing the dimension theory of dynamical systems [1].

The problem of estimating the Hausdorff dimension of invariant subsets has been studied in several papers for different classes of dynamical system (a good overview can be found in [1], [2]).
In [2], for some noninjective skew product maps defined on a square without finite Markov partition, the author calculated Hausdorff dimension of attractor under some conditions. In generally the skew product map $F : Q \rightarrow Q$ with $Q = [a,b] \times [0,1]$ is defined by $F(x,y) = (T(x), g(x,y))$,

where $F : [a,b] \rightarrow [a,b]$ is piecewise monotonic and the map $y \mapsto g(x,y)$ is a contraction on $[0,1]$ for all $x \in [a,b]$. In [2], it was proved that for the skew product map $F(x,y) = (T(x), \varphi(x) + \lambda(y - \frac{1}{2}))$, Hausdorff dimension of the attractor $\Delta := \bigcap_{n=0}^{\infty} F^n(Q)$ of F is $\dim_H(\Lambda) = 1 + \dfrac{h_{top}(T)}{-\log \lambda}$ under some conditions, where $0 < \lambda < \min \{\frac{1}{2}, \inf \frac{1}{|T|}\}$ and $h_{top}(T)$ is the topological entropy of T.

On the other hand Baker's map which can be considered as a special form of the skew product map is well known as a simple example of a "chaotic" dynamical system. When it comes to dimensional theoretical study for this map, in [4] it was considered that, for parameters in some domain, Baker's transformation has ergodic measures with full Hausdorff dimension in its invariant sets but, for parameters in some other domain, variational principle for Hausdorff dimension doesn't hold, that is, there is not a sequence of ergodic measures of which Hausdorff dimension approximates the Hausdorff dimension of the invariant set. In [3] more improved results than one in [4] on the dimensional theoretical properties of the generalized Baker's map were obtained.

According to our search on prior works, it seems that there is no result considered for the existence problem of ergodic measure with full Hausdorff dimension or for the problem on variational principle for Hausdorff dimension for the skew product map yet, whereas such work has done for the Baker's map. And in [2] rather strong condition including

continuously differentiability for the map T and φ was supposed. When it comes to coupled-expanding property, strictly coupled-expanding map implies "chaos" in several senses (see [8] and [9]).

From the above mentioned results, in this paper we study on dimensional theoretical properties for a kind of skew product maps $F(x,y) = (T(x), \varphi(x) + \lambda(y - \frac{1}{2}))$ where T is a coupled-expanding map. We estimate Hausdorff dimension of the attractor of the skew product map F (in section 3) and prove that there exists ergodic measure with full Hausdorff dimension in the attractor for the map F (in section 4).
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2. BASIC NOTATION

Let X be a metric space. Recall that Hausdorff dimension of $A \subset X$ is given by

$$\dim_H(A) = \inf\{s : H^s(A) = 0\} = \sup\{s; H^s(A) = \infty\}$$

where $H^s(A)$ denotes the s-dimensional Hausdorff measure, i.e.

$$H^s(A) = \lim_{\varepsilon \to 0} \inf\left\{\sum |U_i|^s : A \subset \bigcup U_i, |U_i| < \varepsilon\right\},$$

and $|U_i|$ denotes the diameter of a covering element U_i (see [5]).

Let N be a positive integer and $S_i : X \to X$ ($1 \le i \le N$) be a contraction similitude with contraction ratio c_i ($0 < c_i < 1$), i.e. $d(S_i(x), S_i(y)) = c_i d(x,y)$ for any $x, y \in X$. A set $K \subset X$ is said to be **invariant set** with respect to $S := \{S_1, ..., S_N\}$ if $K = \bigcup_{i=1}^{N} S_i(K)$. If there exists a finite set $S = \{S_1, ..., S_N\}$ of contraction similitudes with contraction ratio $c_i (0 < c_i < 1)$ such that K is a invariant set with respect to $S = \{S_1, ..., S_N\}$, and if $H^\alpha(K) > 0$, $H^\alpha(K_i \cap K_j) = 0$ ($i \ne j$) for $\alpha > 0$ such that $\sum_{i=1}^{N} c_i^\alpha = 1$, then K is said to be self-similar set, where H^α is α-dimensional Hausdorff measure and $K_i = S_i(K)$ (see [6]).

Let $D \subset X$ and $f : D \subset X \to X$. If there exists $m \ge 2$ subsets $V_i(1 \le i \le m)$ of D with $V_i \cap V_j = \phi$ and $d(V_i, V_j) > 0$ for all $1 \le i \ne j \le m$, such that

$$f(V_i) \supset \bigcup_{j=1}^{m} V_j, \quad 1 \le i \le m,$$

then f is said to be coupled-expanding map in $V_i, 1 \le i \le m$ (see [8]).

3. HAUSDORFF DIMENSION OF ATTRACTOR OF A SKEW PRODUCT MAP WITH COUPLED-EXPANDING PROPERTY

Theorem 3.1. Let $[a,b] \subset$ be an interval. Suppose that $T : [a,b] \to [a,b]$ is monotonic on $[a,c] \cup [d,b]$ ($a < c < d < b$) and satisfies $T([a,c]) = T([d,b]) = [a,b]$ and $T((a,c)) \subset \{a,b\}$. Assume that $0 < \lambda < 1/2$ and that $\varphi : [a,b] \to [1/2, 1 - \lambda/2]$ is strictly monotonic and satisfies $|\varphi(c) - \varphi(d)| > \lambda$. Then the attractor $\Lambda := \bigcap_{n=0}^{\infty} F^n(Q)$ of the map F definded by $F : Q \to Q$, $F(x,y) = (T(x), \varphi(x) + \lambda(y - \frac{1}{2}))$, $Q = [a,b] \times [0,1]$, has Hausdorff dimension

$$\dim_H(\Lambda) = 1 + \frac{\log 2}{-\log \lambda}. \tag{1}$$

(Remark 1) From the conditions of Theorem 3.1, the map $T : [a,b] \to [a,b]$ is strictly expanding map (see [8]) and therefore the entropy of T is $\log 2$ (see [9]). So we can see that above expression (1) is analogue to the one of [2].

Proof of Theorem 3.1. For convenience of expressing, define follows;

$$V_1 := [a,c], V_2 := [d,b], \Sigma_2^+ := \{1,2\}^{\cup\{0\}},$$

$$\Delta_i := V_i \times [0,1] \quad (i = 1, 2).$$

And set

$$S_{i_0 \ldots i_{p-1}} := F^p(\Delta_{i_{p-1}} \cap \ldots \cap F^{-(p-1)}\Delta_{i_0})$$
$$(i_j \in \{1, 2\}, j \in \{0, \ldots, p-1\}).$$

Then, for any $p \in$, it follows that $S_{i_0 \ldots i_p} \subset S_{i_0 \ldots i_{p-1}}$ since

$$S_{i_0 \ldots i_p} = F^{p+1}(\Delta_{i_p} \cap \ldots \cap F^{-p}\Delta_{i_0}) \subset$$

$$\subset F^p(F\Delta_{i_p}) \cap \ldots \cap F^{-(p-1)}\Delta_{i_0}) \subset S_{i_0 \ldots i_{p-1}}.$$

On the other hand, we have

$$S_{i_0 \ldots i_{p-1}} = F^p((V_{i_{p-1}} \cap \ldots \cap T^{-(p-1)}V_{i_0}) \times [0,1])$$

from

$$\Delta_{i_{p-1}} \cap ... \cap F^{-(p-1)}\Delta_{i_0} = (V_{i_{p-1}} \cap ... \cap T^{-(p-1)}V_{i_0}) \times [0,1] \quad .$$

From the definition of the map F, it follows that

$$F^p(x,y) = (T^p x, \varphi(T^{p-1}x) + \lambda\varphi(T^{p-2}x) + ...$$

$$... + \lambda^{p-1}\varphi(x) + \lambda^p y - \frac{1}{2}(\lambda^p + ... + \lambda)).$$

Since $T^p(V_{i_{p-1}} \cap ... \cap T^{-(p-1)}V_{i_0}) = [a,b]$, the set $S_{i_0...i_{p-1}}$

forms a band with bandwidth λ^p across the lines $x=a$ and $x=b$. Therefore $S_{i_0...i_p...}$ is a piece of the curve across the lines $x=a$ and $x=b$. Then the set Λ consists of uncountable pieces of curve, that is,

$$\Lambda = \bigcup\{S_{i_0...i_p...} : I = i_0...i_p... \in \Sigma_2^+\}.$$

Next, let's consider the skew product map F_u defined by

$$F_u(x,y) = (T(x), \ q_i + \lambda(y - \frac{1}{2})), \ x \in V_i, \ i=1,2,$$

where q_i is defined by $q_1 = \varphi(a)$ and $q_2 = \varphi(b)$.

With the same way as we have considered for F above, we can consider the construction of the attractor of $F_u : Q \to Q$ by $\Lambda_u = \bigcap\limits_{n=0}^{\infty} F_u^n Q$. For $I \in \Sigma_2^+$, let S_I^u be the set for F_u corresponding to S_I. Then we can see that the set S_I^u is a segment with length $b-a$.

Now let's consider the map $\Phi_u : \Lambda_u \to \Lambda$ defined by $(t,y) \in S_I^u \mapsto (t,y') \in S_I$. The map Φ_u maps the point at which S_I^u intersects the line $x=t$, to the point at which S_I intersects the line $x=t$, where $t \in [a,b]$. Then Φ is Lipschitz towards y-ax direction, that is, for any fixing $x \in (a,b)$ we have

$$|\Phi_u(x,y) - \Phi_u(x,y')| \leq |y - y'|.$$

Therefore

$$\dim_H \Lambda \leq \dim_H \Lambda_u$$

by using the result of [7](see Fig.1).

Let $C_u \subset [0,1]$ be the intersection of Λ_u and line $x=t$. Then C_u is self-similar set with respect to the contraction

similitudes with contraction ratio λ and we have $\Lambda_u = [a,b] \times C_u$ (see [6]). Since $\dim_H C_u = \dfrac{\log 2}{-\log \lambda}$, we have
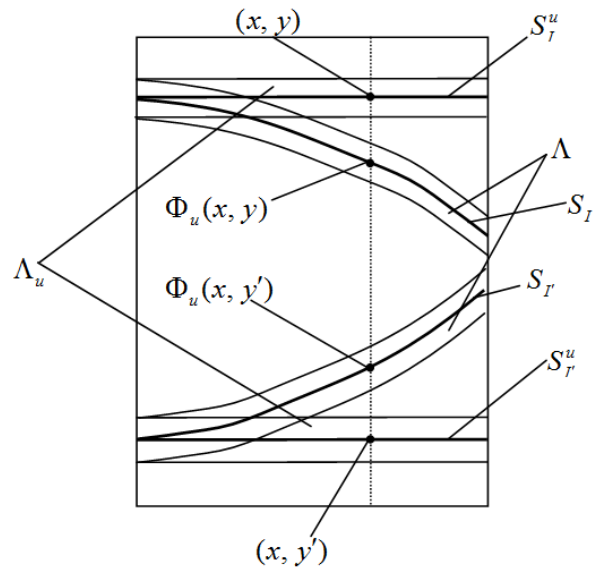

Fig. 1. Image of the point $(x,y) \in \Lambda_u = 1$ **by** Φ_u

$$\dim_H \Lambda_u = 1 + \frac{\log 2}{-\log \lambda}.$$

This implies that

$$\dim_H \Lambda \leq 1 + \frac{\log 2}{-\log \lambda}.$$

To prove the opposite inequality, let's consider the map defined by

$$F_l : Q \to Q, \ F_l(x,y) = \left(T(x), \gamma_i + \lambda(y - \frac{1}{2})\right), \ x \in V_i,$$

where $\gamma_1 = \varphi(c)$ and $\gamma_2 = \varphi(d)$.

By the same procedure as we just have done for F and F_u, we can consider the construction of the attractor Λ_l for F_l. Let S_I^l be the set corresponding to S_I, then S_I^l is also a segment with length $b-a$. Let $\Phi_l : \Lambda \to \Lambda_l$ be a map which maps intersecting point of the line $x=t$ and S_I to intersecting point of the line $x=t$ and S_I^l. Then Φ_l is also Lipshitz towards y-ax direction like as Φ_u (see Fig.2). Therefore

$$\dim_H \Lambda_l \leq \dim_H \Lambda.$$

Let C_l be the intersection of the line $x=t$ and Λ_l. Then C_l is also self-similar set with respect to the contraction similitudes with contraction ratio λ. This means that $\dim_H C_l = \dfrac{\log 2}{-\log \lambda}$. Since $\Lambda_l = [a,b] \times C_l$, we have

$$\dim_H \Lambda \geq 1 + \frac{\log 2}{-\log \lambda} \ . \qquad (2)$$

From (1) and (2) it follows that

$$\dim_H \Lambda = 1 + \frac{\log 2}{-\log \lambda} \ .$$
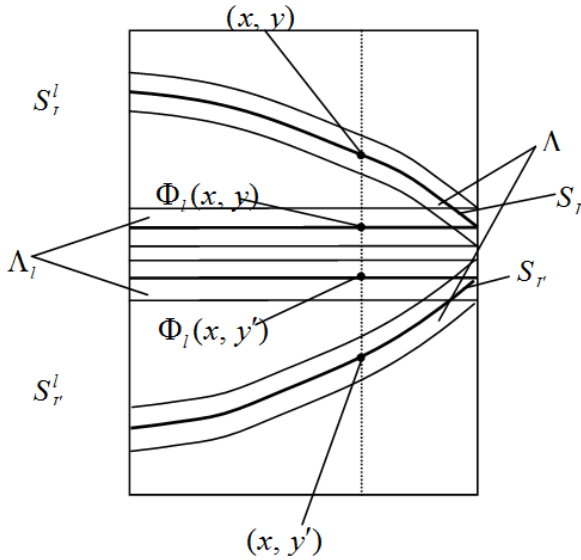
The proof is thus complete.

Fig. 2. Image of the point $(x, y) \in \Lambda$ **by** Φ_l

4. THE EXISTENCE OF ERGODIC MEASURE WITH FULL HAUSDORFF DIMENSION ON ATTRACTOR

Let $M(X, f)$ be the space of all $f -$ invariant and ergodic measures on X.

Proposition 4.1. Suppose that the dynamical system (Λ_2, g) is topologically semi-conjugate to (Λ_1, f), that is, there is a surjection $\varphi : \Lambda_2 \to \Lambda_1$ continuous such that $f \circ \varphi = \varphi \circ g$. Then, If μ is $g -$ invariant and ergodic measure on Λ_2 then $\mu \circ \varphi^{-1}$ is $f -$ invariant and ergodic measure on Λ_1.

Proof. Assume that $f^{-1}(E) = E, E \subset \Lambda_1$. Then $\varphi^{-1}(f^{-1}(E)) = \varphi^{-1}(E)$ and $g^{-1}(\varphi^{-1}(E)) = \varphi^{-1}(E)$ because $f \circ \varphi = \varphi \circ g$. Since μ is $g -$ ergodic, $\mu(\varphi^{-1}(E)) = 0$ or 1. Therefore $\mu \circ \varphi^{-1}$ is $f -$ ergodic measure on Λ_1. And let $A \subset \Lambda_1$ be a measurable set. Since μ is $g -$ invariant, we have

$$\mu \circ \varphi^{-1}(f^{-1}(A)) = \mu(\varphi^{-1}(f^{-1}(A))) =$$
$$= \mu(g^{-1}(\varphi^{-1}(A))) = \mu(\varphi^{-1}(A)) = \mu \circ \varphi^{-1}(A).$$

Therefore the proof is complete.

Theorem 4.1. Suppose that the map

$$F : Q \to Q, \ F(x, y) = \left(T(x), \varphi(x) + \lambda \left(y - \frac{1}{2} \right) \right)$$

satisfies the assumptions of Theorem 3.1 and that T is linear on intervals $[a, c]$ and $[d, b]$. Then there exists a $F -$ invariant and ergodic measure with full dimension on the attractor $\Lambda = \overset{\infty}{\underset{n=0}{\bigcap}} F^n Q$.

Proof. Let L be a normalized Lebesgue measure on interval $[a, b]$. Since C_l is self-similar set with respect to two self-similar contractions with contraction ratio λ, it is homeomorphism to Σ_2^+.

Let β be a Bernoulli measure on Σ_2^+ and μ_β be a measure corresponding to β on C_l. Define μ_l as follows;

$$\mu_l = L \times \mu_\beta .$$

Then μ_l is $F_l -$ ergodic measure defined on Λ_l and it is obvious that

$$\dim_H \mu_l = 1 + \frac{\log 2}{-\log \lambda}$$

by composing of μ_l. By using proposition 4.1, it follows that $\mu := \mu_l \circ \Phi_l$ is $F -$ ergodic measure. Let $\mu(E) = 1, E \subset \Lambda$. From Theorem 3.1 it follows that

$$\dim_H E \leq \dim_H \Lambda = 1 + \frac{\log 2}{-\log \lambda} \ .$$

Since $\mu_l(\Phi_l E) = 1$, from the definition of Hausdorff dimension we have

$$\dim_H \Phi_l E \geq \dim_H \mu_l = 1 + \frac{\log 2}{-\log \lambda} \ .$$

Remember Φ_l is Lipshitz towards y-ax direction, then we have

$$\dim_H E \geq \dim_H \Phi_l E \geq 1 + \frac{\log 2}{-\log \lambda} \ .$$

Thus it follows that

$$\dim_H E = 1 + \frac{\log 2}{-\log \lambda} \ .$$

Considering

$$\dim_H \mu = \inf \left\{ \dim_H E : \mu(E) = 1, E \subset \Lambda \right\},$$

We have

$$\dim_H \mu = 1 + \frac{\log 2}{-\log \lambda} = \dim_H \Lambda \ .$$

The proof is thus complete.

5. CONCLUSION AND FURTHER STUDY

In this paper, for the skew product maps with coupled-expanding property, we estimated Hausdorff dimension of its attractor and proved that there exists an ergodic measure with full Hausdorff dimension for the attractor. In further study we are going to be weaker the limit for the skew product map F which is actually assumed to avoid overlapping of its image here.

6. ACKNOWLEDGMENT

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