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Parameterized KAM Theorem for Differentiable Hamiltonian Vector Fields without Action-Angle Variables

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ABSTRACT
We proved a parameterized KAM theorem in Hamiltonian system which has differentiable Hamiltonian without action-angle coordinates. It is a generalization of the result of [20] that deals with real analytic Hamiltonians.

Keywords
KAM theorem, parameterized KAM theorem, invariant tori, differentiable Hamiltonian, action-angle Variable

1. INTRODUCTION
In this paper, we have presented a KAM theorem on existence of invariant tori with a Diophantine vector for differentiable Hamiltonian vector fields with parameters. We deal with differentiable Hamiltonian vector field which does not require to be a perturbation of an integrable one or to be written in action-angle variables.

The existence problem of invariant tori for Hamiltonian system is often appeared in various scientific fields ([6], [8], [21], [13], [5], [9], [26] etc). Kolmogorov ([18]) has proposed the procedure which clarifies the existence of invariant tori for perturbed real analytic Hamiltonian vector field with action-angle variables at first and Arnold ([1]) has given the rigorous proof based on Kolmogorov's procedure. Moser ([22]) has proved the existence of invariant tori for real analytic area-preserving twist mappings on 2-dimensional annulus with action-angle variables and moreover he has relaxed the assumption of alalyticity of the map to C^{333} differentiability. After that Rüssmann ([24]) has relaxed the differentiability condition with the existence of invariant tori in Hamiltonian system to C^5 class and Takens ([25]) has clarified that it is not enough for C^1 class. Finally Herman ([14]) has clarified that it is enough for C^3 class but not to C^2 -mappings whose second derivatives belong to the Hölder class $C^{1-\delta}$ where $\delta > 0$ is small (see [3], pp.21). However these results are for the Hamiltonian systems to be a perturbation of an integrable one and to be written in action-angle variables. On the other hands, action-angle variables have singularity at elliptic fixed points or in neighborhood of separatrix and the use of action-angle variables are too restrictive in the case of numerical analysis. Moreover, in many practical applications, we have to consider the system which is not near to integrable one but has approximate invariant tori with sufficiently small error ([20]). Llave et al. ([20]) proved the existence of invariant tori for real analytic Hamiltonian system which is not a perturbed integrable one or is written in action-angle variables and Haro ahd Llave ([11], [12]) applied this result in the numerical computation of invariant tori.

The parameterized KAM theory gives an approach to KAM theory of lower dimensional invariant tori in Hamiltonian systems ([3] pp. 44). Broer et al. ([2], [4]), Hoo ([15]) and Huitema ([16]) have got parameterized KAM theorem for Hamiltonian system and Broer et al. ([2], [4], Ciocci et al. ([7]), Hoo ([15]), and Huitema ([16]) have got parameterized KAM theorem for dissipative system. Meanwhile Li and Yi ([19]) have proved a parameterized KAM theorem for the generalized Hamiltonian system (Poisson system). Llave et al. ([20]) have established a parameterized KAM theorem for Hamiltonian fields which is not a perturbed integrable one or is written in action-angle variables.

In this paper, we proved a parameterized KAM theorem for differentiable Hamiltonian vector fields which is not perturbed integrable one or is not written in action-angle variables.

2. PARAMETERIZED KAM THEOREM IN THE CASE OF REAL ANALYTIC HAMILTONIAN

Let n be a positive integer number. We let $|z|=\max_{1\le j\le n}|z_j|$ for $z\in \mathbf{C}^n$ and $|A|=\max_{\substack{1\le i\le m\\ i\le j\le n}}|a_{ij}|$ for $m\times n$ matrix $A=(a_{ij})$.

Let U_ρ denote the complex closed strip of width $\rho>0$: $U_\rho=\{\theta\in \mathbf{C}^n\ ;\ |Im\,\theta|\le\rho\}$. Let B be a compact subset of $\mathbf{R}^n$ which is included in closure of its interior i.e. $B\subset \overline{int\,B}$. Given a function $g\in C^m(B)$, for $m\in \mathbf{Z}_+=\mathbf{N}\cup\{0\}$ we will denote the C^m-norm of g on B by $|g|_{C^m,B}$. Given a 1-periodic function ψ, continuous on n-dimensional tori $\mathbf{T}^n=\mathbf{R}^n/\mathbf{Z}^n$, we denote the average of ψ on $\mathbf{T}^n$ by

$$<\psi>=\int_{\mathbf{T}^n}\psi(\theta)d\theta .$$

Definition 1. Given $\gamma>0$ and $\sigma>n-1$, we define $D(\gamma,\sigma)$ as the set of frequency vectors $\omega=(\omega_1,\ \cdots,\ \omega_n)\in \mathbf{R}^n$ satisfying the *Diophantine condition*:

$$|k\cdot\omega|\ge\gamma|k|_1^{-\sigma},\ \forall k\in \mathbf{Z}^n-\{0\}$$

where $|k|_1=|k_1|+\cdots+|k_n|$.

Definition 2. If V is an open subset of topological space of X then we denote this fact by $V\overset{\circ}{\subset}X$. We will assume that $\mathbf{U}$ is either $\mathbf{T}^n\times U$ with $U\overset{\circ}{\subset}\mathbf{R}^n$ or $\mathbf{U}\overset{\circ}{\subset}\mathbf{R}^{2n}$. Let Q is a subset of $\mathbf{R}^d$ $(d\in \mathbf{N})$ which is included in closure of its interior. We suppose that $H:\mathbf{U}\times Q\to \mathbf{R}$ satisfy following two conditions:

1) for each $x\in\mathbf{U}$ the function $H(x,\ \cdot\ ):Q\to\mathbf{R}$ is C^2,

2) for each $\lambda\in Q$ the function $H_\lambda:=H(\cdot,\ \lambda):\mathbf{U}\to\mathbf{R}$ is real analytic Hamiltonian function.

Then we call $H=H(x,\ \lambda)$ as d *-parametric family of Hamiltonian functions*.

We denote *n*-dimensional unit matrix as I. Let $J=\begin{pmatrix} O & I\\ -I & O\end{pmatrix}$. We will consider the first-order partial differential equation

$$\partial_\omega K(\theta)=J\nabla H_\lambda(K(\theta)) \qquad (1)$$

for $2n$ -parametric family of Hamiltonian functions $H:\mathbf{U}\times Q\to\mathbf{R}$, where unknown is $K:\mathbf{T}^n\to\mathbf{U}$. We call equation (1) as *Hamiltonian invariant torus equation*. In this place, $\omega\in D(\gamma,\ \sigma)\subset\mathbf{R}^n$ and ∂_ω is the derivative in direction ω:

$$\partial_\omega K=\sum_{i=1}^n\omega_i\frac{\partial}{\partial\theta_i}K .$$

Definition 3. Suppose that $H:\mathbf{U}\times Q\to\mathbf{R}$ is a $2n$ -parametric family of Hamiltonian functions and $\omega\in D(\gamma,\ \sigma)\subset\mathbf{R}^n$. We denote as Π_ρ the Banach space of maps $K:U_\rho\to\mathbf{U}$ which are 1-periodic in all its variables, real analytic on the interior of U_ρ and continuous on the boundary of U_ρ with norm $\|K\|_\rho=\sup_{\theta\in U_\rho}|K(\theta)|$. Let $\lambda_0\in Q$. We suppose that $K\in\Pi_\rho$ satisfy following two conditions:

1) There exists a $n\times n$ matrix-valued function $N(\theta)$ satisfying

$$N(\theta)(DK(\theta)^T DK(\theta))=I ,$$

2) $<\Lambda_0>$ is invertible with

$$\Lambda_0(\theta)=\begin{pmatrix} N_0(\theta)DK_0(\theta)^T J\left(\frac{\partial}{\partial\lambda}\nabla H_\lambda(K_0(\theta))\Big|_{\lambda=\lambda_0}\right)\\ DK_0(\theta)^T J\left(\frac{\partial}{\partial\lambda}\nabla H_\lambda(K_0(\theta))\Big|_{\lambda=\lambda_0}\right)\end{pmatrix}.$$

Then we call K is *non-degenerate*.

Theorem. (Theorem 3 in [20]) Let $\omega\in D(\gamma,\ \sigma)$. Assume that $K_0\in\Pi_\rho$ is non-degenerate. Assume that $H:\mathbf{U}\times Q\to\mathbf{R}$ is a $2n$ -parametric family of Hamiltonian functions and there exist $r>0$ such that for any $\lambda\in Q$, H_λ is can be holomorphically extended to r -neighbor-hood of the image of U_ρ under K_0:

$$B_r=B_r(K_0)=\{z\in\mathbf{R}^n\ ;\ \inf_{\theta\in U_\rho}|z-K_0(\theta)|<r\} .$$

Let's suppose that Q is a compact subset of $\mathbf{R}^n$ which is included in closure of its interior and involve r -neighborhood of $\lambda_0\in\mathbf{R}^{2n}$. Define the error function for λ_0, K_0 by

$$e_0(\theta)=J\nabla H_{\lambda_0}(K_0(\theta))-\partial_\omega K_0(\theta)) .$$

Let $\delta_0=min(1,\ \rho/12)$. Then there exists a constant $c>0$, depending on σ, n, r, ρ, $|H|_{C^3,B_r\times Q}$, $\|DK_0\|_\rho$, $\|N_0\|_\rho$, $|<\Lambda_0>^{-1}|$ such that if

$$c\gamma^{-4}\delta_0^{-4\sigma}\|e_0\|_\rho<1 \qquad (2)$$

$$c\gamma^{-2}\delta_0^{-2\sigma}\|e_0\|_\rho<r \qquad (3)$$

then there exist $\lambda_\infty \in Q$ and $K_\infty \in \Pi_{\rho/2}$ which satisfies the non-degenerate conditions such that

$$\partial_\omega K_\infty(\theta) = J\nabla H_{\lambda_\infty}(K_\infty(\theta)).$$

Moreover λ_∞ and K_∞ satisfy

$$||K_\infty - K_0||_{\rho/2} \le r \quad (4)$$

$$|\lambda_\infty - \lambda_0| < r \quad (5)$$

and $||DK_\infty||_{\rho/2}$, $||N_\infty||_{\rho/2}$, $|<\Lambda_\infty>^{-1}|$ satisfy

$$||DK_\infty||_{\rho/2} \le ||DK_0||_{\rho_0} + \beta,$$

$$||N_\infty||_{\rho/2} \le ||N_0||_{\rho_0} + \beta,$$

$$|<\Lambda_\infty>^{-1}| \le |<\Lambda_0>^{-1}| + \beta.$$

In this place, N_∞ and Λ_∞ are as in definition 3, replacing K with K_∞ and λ with λ_∞ and $\beta = \gamma^{-2}\delta_0^{2\sigma-1}2^{-4\sigma}$.

Remark 1. The dependence of constant c on $|H|_{C^3, Ç\$\times Q}$, $||DK_0||_\rho$, $||N_0||_\rho$, $|<\Lambda_0>^{-1}|$ is polynomial. That is, there exists a polynomial, $\lambda(y_1, y_2, y_3, y_4)$ with positive coefficients depending on σ, n and such that

$$c = \lambda(|H|_{C^3, Ç\$\times Q}, ||DK_0||_\rho, ||N_0||_\rho, |<\Lambda_0>^{-1}|)$$

(Remark 15 in [20]).

3. PARAMETERIZED KAM THEOREM IN THE CASE OF DIFFERENTIABLE HAMILTONIAN

To prove the existence of invariant tori in the case of differentiable Hamiltonian with parameter, we need some approximation propositions.

Lemma 1. Let $E_n = [a_1, b_1]\times[a_2, b_2]\times\cdots\times[a_n, b_n]$ and $f \in C^4(E_n)$. Then there exists a sequence of real analytic functions $\{f_k\}$ on E_n such that

$$|f_k - f|_{C^3, E_n} \to 0, \ k \to \infty.$$

proof. See [17]. ▯

Lemma 2. Let $l \in \mathbf{N}$. If a sequence of functions $\{f_k(x)\}$ real analytic on $U_{\rho/4^{k-1}}$ satisfies the following inequality

$$||f_k(x) - f_{k+1}(x)||_{\rho/4^k} \le A(4^{-l})^k,$$

where $A \ge 0$ is a suitable constant, then $f_k(x)$ converges to certain function $f \in C^l(\mathbf{T}^n)$.

proof. See lemma 1 of Chapter 3, Section 7 in [23]. ▯

Our main result is as follows:

Theorem 1. Let $\omega \in D(\gamma, \sigma)$ ($\sigma > n-1$). Assume that $K_0 \in \Pi_\rho$ and K_0 is non-degenerate. Assume that $H: \mathbf{U}\times Q \to \mathbf{R}$ is C^l ($l \ge 4$) and that is can be extended to $B_{3r}(K_0)\times A(Q)$ as C^l class. In this place, the parameter set Q is a compact subset of $\mathbf{R}^n$ which is included in closure of its interior and include $2r$-neighbor of λ_0 and $A(Q)$ is a some rectangle region involving Q. Define the error function

$$e_0(\theta) = J\nabla H_{\lambda_0}(K_0(\theta)) - \partial_\omega K_0(\theta) \ (\theta \in U_\rho)$$

and $\delta_0 = min(1, \rho/12)$. For a certain constant $c > 0$, depending on σ, n, $|H|_{C^3, B_{2r}\times Q}$, $||DK_0||_\rho$, $||N_0||_\rho$, $|<\Lambda_0>^{-1}|$,
if error function e_0 satisfies (2),(3) then there exist $\lambda_\infty \in Q$ and C^1 map $K_\infty: \mathbf{T}^n \to \mathbf{U}$ satisfying

$$\partial_\omega K_\infty(\theta) = J\nabla H_{\lambda_\infty}(K_\infty(\theta)), \ (\theta \in \mathbf{T}^n).$$

Proof. We use the following notations:

$$\mu_0 = |H|_{C^3, B_{2r}\times Q}, \ d_0 = ||DK_0||_\rho, \ \nu_0 = ||N_0||_\rho,$$

$$\tau_0 = |<\Lambda_0>^{-1}|, \ \beta = \gamma^{-2}\delta_0^{2\sigma-1}\frac{1}{2^{4\sigma} - 2^{2\sigma+1}},$$

$$\mu = \mu_0 + 1, \ d = d_0 + \beta, \ \nu = \nu_0 + \beta, \ \tau = \tau_0 + \beta + 1.$$

Let $c = \lambda(\mu, d, \nu, \tau)$. Since $B_{3r}(K_0)$ is a bounded subset of $\mathbf{R}^n$, there exists a rectangle region

$$E_{2n}(K_0) = [a_1, b_1]\times\cdots\times[a_{2n}, b_{2n}]$$

satisfying

$$B_{3r}(K_0) \subset E_{2n}(K_0).$$

Since $B_{5r/2}(K_0)$ is a open neighborhood of $K_0(U_\rho)$, there exists a function $\varphi \in C^\infty(E_{2n}(K_0))$ such that

$$\varphi(z) = \begin{cases} 1, & z \in K_0(U_\rho) \\ 0, & z \in E_{2n}(K_0) \setminus B_{5r/2}(K_0) \end{cases}$$

and

$$0 \le \varphi(z) \le 1, \ (z \in E_{2n}(K_0)).$$

Now define the function $\psi: E_{2n}(K_0)\times A(Q) \to \mathbf{R}$ by $\psi := \varphi \circ p_1$, where $p_1: E_{2n}(K_0)\times A(Q) \to E_{2n}(K_0)$ is projection map. Since projection map is C^∞, ψ is C^∞. Now we extend the function H onto $E_{2n}(K_0)\times A(Q)$ arbitrarily and consider the function

$$H\cdot\psi: E_{2n}(K_0)\times A(Q) \to \mathbf{R}.$$

Because H and φ are both C^l and $\varphi(z) = 0, (z \in (E_{2n}(K_0) \setminus B_{5r/2}(K_0))\times A(Q))$, $H\cdot\psi$ is C^l on $(E_{2n}(K_0) \setminus B_{5r/2}(K_0))\times A(Q)$. Therefore $H\cdot\varphi$ is C^l on $E_{2n}(K_0)\times A(Q)$. And $H\cdot\varphi$ preserves function values of H on $B_{2r}(K_0)\times A(Q)$. So we denote $H\cdot\psi$ as H. Then from the lemma 1, there exists a sequence of real analytic functions on $E_{2n}(K_0)\times A(Q)$ which C^3-converges to H. If we take appropriate subsequence of the sequence then we get a sequence of real analytic functions on $E_{2n}(K_0)\times A(Q)$, $\{H^k\}_{k\geq 1}$ such that

$$|H^k - H^{k+1}|_{C^3, E_{2n}(K_0)\times A(Q)} \leq A(4^{-k})^{l+2\sigma} \quad (6)$$

$$|H^k - H|_{C^3, E_{2n}(K_0)\times A(Q)} \leq A(4^{-k})^{l+2\sigma} \quad (7)$$

where $A\geq 0$ is a constant depending on only $|H|_{C^3, E_{2n}(K_0)\times A(Q)}$.

Because $B_{2r}(K_0)\subset E_{2n}(K_0)$, we obtain that

$$|H^k - H^{k+1}|_{C^3, B_{2r}(K_0)\times A(Q)} \leq A(4^{-k})^{l+2\sigma},$$
$$|H^k - H|_{C^3, B_{2r}(K_0)\times A(Q)} \leq A(4^{-k})^{l+2\sigma}.$$

Let us prove the theorem 1 in two steps.

First Step. For the first step, we will prove that for some number k_0 and $\lambda_{k_0}\in Q$, there exists a solution $K_{k_0}\in \Pi_{\rho/2}$ of (1) with Hamiltonian $H^{k_0}_{\lambda_{k_0}}$ satisfying

$$\|K_{k_0} - K_0\|_{\rho/2} \leq r,$$
$$|\lambda_{k_0} - \lambda_0| \leq r.$$

For this, let us prove that for sufficient large number k_0, for some $\lambda_{k_0}\in Q$, the pair $(H^{k_0}_{\lambda_{k_0}}, K_0)$ satisfy the assumptions of the theorem in section 2, i.e. c_{k_0} and e_{k_0} defined by

$$c_{k_0} = \lambda(|H^{k_0}|_{C^3, B_{2r}(K_0)\times Q}, \|DK_0\|_\rho, \|N_0\|_\rho, |<\Lambda_0^{k_0}>^{-1}|),$$

$$e_{k_0} = J\nabla H^{k_0}_{\lambda_0}(K_0(\theta)) - \partial_\omega K_0(\theta)$$

satisfy (2) and (3). First we prove that $c_{k_0} < c$. Performing some simple computations, we obtain

$$\Lambda_0^{k_0}(\theta) = \begin{pmatrix} N_0(\theta)DK_0(\theta)^T J\left(\frac{\partial}{\partial\lambda}\nabla H^{k_0}_\lambda(K_0(\theta))\Big|_{\lambda=\lambda_0}\right) \\ DK_0(\theta)^T J\left(\frac{\partial}{\partial\lambda}\nabla H^{k_0}_\lambda(K_0(\theta))\Big|_{\lambda=\lambda_0}\right)\end{pmatrix} =$$
$$= \begin{pmatrix} N_0(\theta)DK_0(\theta)^T J\left(\frac{\partial}{\partial\lambda}\nabla H_\lambda(K_0(\theta))\Big|_{\lambda=\lambda_0}\right) \\ DK_0(\theta)^T J\left(\frac{\partial}{\partial\lambda}\nabla H_\lambda(K_0(\theta))\Big|_{\lambda=\lambda_0}\right)\end{pmatrix} +$$
$$+ \begin{pmatrix} N_0(\theta)DK_0(\theta)^T J\left(\frac{\partial}{\partial\lambda}\nabla\left(H^{k_0}_\lambda(K_0(\theta)) - H_\lambda(K_0(\theta)\right)\Big|_{\lambda=\lambda_0}\right) \\ DK_0(\theta)^T J\left(\frac{\partial}{\partial\lambda}\nabla\left(H^{k_0}_\lambda(K_0(\theta)) - H_\lambda(K_0(\theta)\right)\Big|_{\lambda=\lambda_0}\right)\end{pmatrix} =$$
$$= \Lambda_0(\theta) + \Psi_0(\theta).$$

Because $\|\Psi_0\|_{\rho_0} \leq d(\nu+1)|H^{k_0} - H|_{C^3, B_{2r}(K_0)\times A(Q)}$ and $|H^{k_0} - H|_{C^3, B_{2r}(K_0)\times A(Q)} \to 0\ (k_0\to\infty)$, for sufficiently large k_0,

$$d(\nu+1)|H^{k_0} - H|_{C^3, B_{2r}(K_0)\times A(Q)}\ \tau < \frac{1}{2}$$

holds. We obtain that

$$|<\Psi_0>| \leq d(\nu+1)|H^{k_0} - H|_{C^3, B_{2r}(K_0)\times A(Q)}$$

And from $|<\Lambda_0>^{-1}| = \tau_0 < \tau$, $|<\Psi_0(\theta)>|\cdot|<\Lambda_0>^{-1}| < \frac{1}{2}$.

Hence $I + <\Lambda_0>^{-1}<\Psi_0>$ is invertible and therefore $<\Lambda_0^{k_0}> = <\Lambda_0>[I + <\Lambda_0>^{-1}<\Psi_0>]$ is invertible. We obtain that

$$<\Lambda_0^{k_0}>^{-1} = (I + <\Lambda_0>^{-1}<\Psi_0>)^{-1}<\Lambda_0>^{-1}.$$

We deform the above equality like that

$$<\Lambda_0^{k_0}>^{-1} = (I + <\Lambda_0>^{-1}<\Psi_0>)^{-1}(I + <\Lambda_0>^{-1}<\Psi_0> - <\Lambda_0>^{-1}<\Psi_0>)<\Lambda_0>^{-1} =$$
$$= <\Lambda_0>^{-1} - (I + <\Lambda_0>^{-1}<\Psi_0>)^{-1}<\Lambda_0>^{-1}<\Psi_0><\Lambda_0>^{-1}.$$

From

$$|(I + <\Lambda_0>^{-1}<\Psi_0>)^{-1}| \leq \sum_{i=0}^{\infty}\left|<\Lambda_0>^{-1}<\Psi_0>\right|^i < 2,$$

following inequality holds:

$$|<\Lambda_0^{k_0}>^{-1}| \leq |<\Lambda_0>^{-1}| + 2d(\nu+1)\tau^2\,|H^{k_0} - H|_{C^3, B_{2r}(K_0)\times A(Q)}.$$

Meanwhile from the triangle inequality

$$|H^{k_0}|_{C^3, B_{2r}(K_0)\times A(Q)} \leq |H|_{C^3, B_{2r}(K_0)\times A(Q)} + |H^{k_0} - H|_{C^3, B_{2r}(K_0)\times A(Q)}$$

holds.

Because the sum $\sum_{k=1}^{\infty}|H^{k+1} - H^k|_{C^3, B_{2r}(K_0)\times A(Q)}$ converges, for sufficiently large k_0, the following inequalities hold:

$$d(\nu+1)|H^k - H|_{C^3, B_{2r}(K_0)\times A(Q)}\ \tau < \frac{1}{4},\ (k\geq k_0) \quad (8)$$

$$|H^k - H|_{C^3, B_{2r}(K_0)\times A(Q)} < 1,\ (k\geq k_0) \quad (9)$$

$$2d(\nu+1)\tau^2\Big(|H^{k_0}-H|_{C^3,B_{2r}(K_0)\times A(Q)}+ + \sum_{k=k_0+1}^{\infty}|H^k-H^{k-1}|_{C^3,B_{2r}(K_0)\times A(Q)}\Big)<1 \quad (10)$$

Hence

$$|H^{k_0}|_{C^3,B_{2r}(K_0)\times A(Q)}\le|H|_{C^3,B_{2r}(K_0)\times A(Q)}+|H^{k_0}-H|_{C^3,B_{2r}(K_0)\times A(Q)}\le|H|_{C^3,B_{2r}(K_0)\times A(Q)}+1,$$

$$|<\Lambda_0^{k_0}>^{-1}|\le\ |<\Lambda_0>^{-1}|+2d(\nu+1)\tau^2\ |H^{k_0}-H|_{C^3,B_{2r}(K_0)\times A(Q)}\le|<\Lambda_0>^{-1}|+1$$

hold and from the definition of c

$$c_{k_0}<c. \quad (11)$$

Let's confirm the assumptions (2),(3) in the theorem in section 2 for $e_{k_0}=J\nabla H_{\lambda_0}^{k_0}(K_0(\theta))-\partial_\omega K_0(\theta)$. We obtain that

$$\begin{aligned}
c_{k_0}\gamma^{-4}\delta_0^{-4\sigma}\ ||e_{k_0}||_\rho &=\\
&=c_{k_0}\gamma^{-4}\delta_0^{-4\sigma}\ ||J\nabla H_{\lambda_0}^{k_0}(K_0(\theta))-\partial_\omega K_0(\theta)||_\rho\le\\
&\le c\gamma^{-4}\delta_0^{-4\sigma}(||J\nabla H_{\lambda_0}^{k_0}(K_0(\theta))-J\nabla H_{\lambda_0}(K_0(\theta))||_\rho\\
&\quad+||J\nabla H_{\lambda_0}(K_0(\theta))-\partial_\omega K_0(\theta)||_\rho)=\\
&=c\gamma^{-4}\delta_0^{-4\sigma}(||J\nabla H_{\lambda_0}^{k_0}(K_0(\theta))-J\nabla H_{\lambda_0}(K_0(\theta))||_\rho\\
&\quad+||e_0||_\rho)\le c\gamma^{-4}\delta_0^{-4\sigma}\ |H^{k_0}-H|_{C^3,B_{2r}(K_0)\times A(Q)}\\
&\quad+c\gamma^{-4}\delta_0^{-4\sigma}\ ||e_0||_\rho .
\end{aligned}$$

From the assumption (2) and

$$|H^{k_0}-H|_{C^3,B_{2r}(K_0)\times A(Q)}\to 0,\ (k_0\to\infty),$$

for sufficiently large k_0, the following inequality holds:

$$c\gamma^{-4}\delta_0^{-4\sigma}\ |H^{k_0}-H|_{C^3,B_{2r}(K_0)\times A(Q)}+c\gamma^{-4}\delta_0^{-4\sigma}\ ||e_0||_\rho<1.$$

Similarly,

$$c_{k_0}\gamma^{-2}\delta_0^{-2\sigma}\ ||e_{k_0}||_\rho\le c\gamma^{-2}\delta_0^{-2\sigma}\ |H^{k_0}-H|_{C^3,B_{2r}(K_0)\times A(Q)}+c\gamma^{-2}\delta_0^{-2\sigma}\ |e_0||_\rho.$$

And from the assumption (3), for sufficiently large k_0,

$$c\gamma^{-2}\delta_0^{-2\sigma}\ |H^{k_0}-H|_{C^3,B_{2r}(K_0)\times A(Q)}+c\gamma^{-2}\delta_0^{-2\sigma}\ |e_0||_\rho<r.$$

Thus we obtain that

$$c_{k_0}\gamma^{-4}\delta_0^{-4\sigma}\ ||e_{k_0}||_\rho<1, \quad (12)$$

$$c_{k_0}\gamma^{-2}\delta_0^{-2\sigma}\ ||e_{k_0}||_\rho<r. \quad (13)$$

Therefore for the pair $(H_{\lambda_0}^{k_0},K_0)$, the assumptions of the theorem in section 2 holds. So there exist $\lambda_{k_0}\in Q$ and a solution $K_{k_0}\in\Pi_{\rho/2}$ of (1) with Hamiltonian $H_{\lambda_0}^{k_0}$ which is non-degenerate. Moreover λ_{k_0} and K_{k_0} satisfy

$$||K_{k_0}-K_0||_{\rho/2}\le c\gamma^{-2}\delta_0^{-2\sigma}\ ||e_0||_\rho\le r, \quad (14)$$

$$|\lambda_{k_0}-\lambda_0|\le r. \quad (15)$$

Now we take the positive integer number $k_0\ge 2$ more sufficiently large in order that k_0 satisfies

$$A(4^{-(k_0-1)})^{l+2\sigma}\le||e_0||_\rho .$$

From (6) for any integer number $k\ge k_0$,

$$|H^k-H^{k-1}|_{C^3,B_{2r}(K_0)\times A(Q)}\le||e_0||_\rho\ (4^{l+2\sigma})^{-k+1}. \quad (16)$$

From now on we consider the subsequence $\{H^k\}_{k=k_0}^{\infty}$ and for simplicity we denote the sequence $\{H^k\}_{k=k_0}^{\infty}$ as $\{H^k\}_{k=1}^{\infty}$.

Second Step. For the second step, we will prove the following statements:

For any $k\in \mathbf{N}$, there exists $\lambda_k\in Q$ and a map $K_k\in\Pi_{\rho/2^k}$ which is a solution of (1) with Hamiltonian $H_{\lambda_k}^k$. And moreover they satisfy

$$||K_k-K_{k-1}||_{\rho/4^k}\le r(\frac{1}{4^{l+\sigma}})^{k-1},$$

$$|\lambda_k-\lambda_{k-1}|\le r(\frac{1}{4^{l+\sigma}})^{k-1}\quad(k\ge 1).$$

We will prove them using induction. We define the following notations:

$$\rho_k=\frac{\rho}{2^{k-1}},\quad \delta_k=\frac{\rho_k}{12},\quad r_k=r(\frac{1}{4^{l+\sigma}})^{k-1},$$

$$B_{r_k}(K_k)=\{z\in\mathbf{R}^n;\ \inf_{\theta\in U_{\rho_k}}|z-K_k(\theta)|<r_k\},$$

$$e_k(\theta)=J\nabla H_{\lambda_k}^k(K_{k-1}(\theta))-\partial_\omega K_{k-1}(\theta),$$

$$\mu_k=|H^k|_{C^3,B_{r_{k-1}}(K_{k-1})\times Q},\ d_k=||DK_{k-1}||_{\rho_{k-1}},$$

$$\nu_k=||N_{k-1}||_{\rho_{k-1}},\ \tau_k=|<\Lambda_{k-1}^k>^{-1}|.$$

And we define

$$c_k=\lambda(\mu_k,\ d_k,\ \nu_k,\ \tau_k).$$

To prove the statement in second step, we will prove the following five statements with $k \geq 1$, using induction:

A1(k) $|\lambda_k - \lambda_0| \leq r \sum_{i=0}^{k-1} (\frac{1}{4^{l+\sigma}})^i$,

A2(k) $\|K_k - K_0\|_{\rho_{k+1}} \leq r \sum_{i=0}^{k-1} (\frac{1}{4^{l+\sigma}})^i \leq r \frac{4^{l+\sigma}}{4^{l+\sigma}-1} \leq \frac{4}{3} r$,

A3(k) $c_k \leq c$,

A4(k) $c_k \gamma^{-4} \delta_k^{-4\sigma} \|e_k\|_{\rho_k} < 1$,

A5(k) $c_k \gamma^{-2} \delta_k^{-2\sigma} \|e_k\|_{\rho_k} < r_k$.

If A1(k)-A5(k) hold then the assumptions of the theorem 1 in section 2 is satisfied for a pair $(H_{\lambda_k}^k,\ K_{k-1})$. Then we can obtain λ_k and K_k from the pair $(H_{\lambda_k}^k,\ K_{k-1})$.

First, consider the case of $k=1$. A1(1) is followed by (15) and A2(1), A3(1), A4(1), A5(1) are respectively followed by (14) , (11), (12), (13) in first step.

Assume that A1(j)-A5(j) hold for $j=1,\cdots,k-1$ and let us prove A1(k)-A5(k). For any $j=2,\cdots,k$, we obtain K_j, a solution of (1) with Hamiltonian $H_{\lambda_j}^j$, by applying the theorem 1 in section 2 to $(H_{\lambda_j}^j, K_{j-1})$. Then inequality (4) implies

$$\|K_k - K_0\|_{\rho_{k+1}} \leq \|K_k - K_{k-1}\|_{\rho_{k+1}} + \|K_{k-1} - K_0\|_{\rho_k} \leq$$
$$\leq r_k + r\sum_{i=0}^{k-2} (\frac{1}{4^{l+\sigma}})^i \leq r \sum_{i=0}^{k-1} (\frac{1}{4^{l+\sigma}})^i$$

and

$$|\lambda_k - \lambda_0| \leq |\lambda_k - \lambda_{k-1}| + |\lambda_{k-1} - \lambda_0| \leq$$
$$\leq r_k + r\sum_{i=0}^{k-2} (\frac{1}{4^{l+\sigma}})^i \leq r \sum_{i=0}^{k-1} (\frac{1}{4^{l+\sigma}})^i .$$

This proves A1(k) and A2(k).

Let's prove A3(k). From the inequality (9),

$$\begin{aligned}\mu_k &= |H^k|_{C^3,\, B_{r_{k-1}}(K_{k-1})\times Q} \leq |H^k|_{C^3, B_{2r}(K_0)\times Q} \\ &\leq |H|_{C^3, B_{2r}(K_0)\times Q} + |H^k - H|_{C^3, B_{2r}(K_0)\times Q} \\ &\leq |H|_{C^3, B_{2r}(K_0)\times Q} + 1 \leq \mu .\end{aligned}$$

And from the statement of theorem 1 in section 2, we obtain

$$\begin{aligned}d_k &= \|DK_{k-1}\|_{\rho_{k-1}} \\ &\leq \|DK_{k-2}\|_{\rho_{k-2}} + \gamma^{-2}\delta_{k-1}^{2\sigma-1} 2^{-4\sigma} \leq \cdots \leq \\ &\leq \|DK_0\|_{\rho_0} + \gamma^{-2}\delta_{k-1}^{2\sigma-1} 2^{-4\sigma} + \cdots + \gamma^{-2}\delta_0^{2\sigma-1}2^{-4\sigma} \\ &\leq \|DK_0\|_{\rho_0} + \sum_{i=0}^{\infty} \gamma^{-2} \delta_i^{2\sigma-1} 2^{-4\sigma} \\ &\leq \|DK_0\|_{\rho_0} + \gamma^{-2}\delta_0^{2\sigma-1}2^{-4\sigma} \sum_{i=0}^{\infty} (1/2^i)^{2\sigma-1} \\ &\leq \|DK_0\|_{\rho_0} + \gamma^{-2}\delta_0^{2\sigma-1}2^{-4\sigma}\frac{1}{1-2^{-2\sigma+1}} \\ &\leq \|DK_0\|_{\rho_0} + \gamma^{-2}\delta_0^{2\sigma-1} \frac{1}{2^{4\sigma} - 2^{2\sigma+1}} \leq \\ &\leq d_0 + \beta = d .\end{aligned}$$

Similarly, we get

$$\nu_k = \|N_{k-1}\|_{\rho_{k-1}} \leq \nu_0 + \beta = \nu .$$

Now let us prove the following inequalities by using induction:

$$\tau_k = |\langle \Lambda_{k-1}^{H_k} \rangle^{-1}| \leq \tau_0 + \beta + 1 = \tau ,$$

$$|\langle \Lambda_{k-1}^{H_{k-1}} \rangle^{-1}| \leq \tau_0 + \beta + 1 = \tau .$$

In the case of $k=1$ is trivial. Assume that they hold for $i=0,\cdots,k-1$.

Performing similar computation in step 1, we obtain

$$\Lambda_{k-1}^k(\theta) = \begin{pmatrix} N_{k-1}(\theta) DK_{k-1}(\theta)^T J\left(\frac{\partial}{\partial\lambda}\nabla H_\lambda^{k-1}(K_{k-1}(\theta))\Big|_{\lambda=\lambda_{k-1}}\right) \\ DK_{k-1}(\theta)^T J\left(\frac{\partial}{\partial\lambda}\nabla H_\lambda^{k-1}(K_{k-1}(\theta))\Big|_{\lambda=\lambda_{k-1}}\right)\end{pmatrix} +$$
$$+\begin{pmatrix} N_{k-1}(\theta) DK_{k-1}(\theta)^T J\left(\frac{\partial}{\partial\lambda}\nabla\left(H_\lambda^k(K_{k-1}(\theta)) - H_\lambda^{k-1}(K_{k-1}(\theta)\right)\Big|_{\lambda=\lambda_{k-1}}\right) \\ DK_{k-1}(\theta)^T J\left(\frac{\partial}{\partial\lambda}\nabla\left(H_\lambda^k(K_{k-1}(\theta)) - H_\lambda^{k-1}(K_{k-1}(\theta)\right)\Big|_{\lambda=\lambda_{k-1}}\right)\end{pmatrix} =$$
$$= \Lambda_{k-1}^{k-1}(\theta) + \Psi_{k-1}(\theta).$$

In this place we get

$$\begin{aligned}\|\Psi_{k-1}(\theta)\|_{\rho_{k-1}} &\leq d_k(\nu_k+1)|H^k - H^{k-1}|_{C^3,\, B_{r_{k-1}}(K_{k-1})\times Q} \\ &\leq d(\nu+1)|H^k - H^{k-1}|_{C^3, B_{2r}(K_0)\times Q} .\end{aligned}$$

From (8), we obtain

$$\begin{aligned}& d(\nu+1)|H^k - H^{k-1}|_{C^3, B_{2r}(K_0)\times Q} \cdot |\langle\Lambda_{k-1}^{k-1}\rangle^{-1}| \\ &< d(\nu+1)|H^k - H^{k-1}|_{C^3, B_{2r}(K_0)\times Q} \cdot \tau \leq \frac{1}{2}\end{aligned} .$$

Thus

$$|\langle \Psi_{k-1}\rangle| \cdot |\langle \Lambda_{k-1}^{k-1}\rangle^{-1}| \leq \frac{1}{2} .$$

Hence $I + \langle\Lambda_{k-1}^{k-1}\rangle^{-1}\langle\Psi_{k-1}\rangle$ is invertible and therefore

$$<\Lambda_{k-1}^{k}> = <\Lambda_{k-1}^{k-1}>[I+<\Lambda_{k-1}^{k-1}>^{-1}<\Psi_{k-1}>]$$

is invertible, i.e.

$$<\Lambda_{k-1}^{k}>^{-1} = [I+<\Lambda_{k-1}^{k-1}>^{-1}<\Psi_{k-1}>]^{-1}<\Lambda_{k-1}^{k-1}>^{-1}.$$

We deform of expression of $<\Lambda_{k-1}^{k}>^{-1}$ like that

$$\begin{aligned}&<\Lambda_{k-1}^{k}>^{-1}=(I+<\Lambda_{k-1}^{k-1}>^{-1}<\Psi_{k-1}>)^{-1}\\ &[I+<\Lambda_{k-1}^{k-1}>^{-1}<\Psi_{k-1}>-<\Lambda_{k-1}^{k-1}>^{-1}<\Psi_{k-1}>]<\Lambda_{k-1}^{k-1}>^{-1}\\ &=<\Lambda_{k-1}^{k-1}>^{-1}-[I+<\Lambda_{k-1}^{k-1}>^{-1}<\Psi_{k-1}>]^{-1}<\Lambda_{k-1}^{k-1}>^{-1}\\ &<\Psi_{k-1}><\Lambda_{k-1}^{k-1}>^{-1}.\end{aligned}$$

Since

$$|(I+<\Lambda_{k-1}^{k-1}>^{-1}<\Psi_{k-1}>)^{-1}| \le \sum_{i=0}^{\infty}|<\Lambda_{k-1}^{k-1}>^{-1}<\Psi_{k-1}>|^{i}<2,$$

we get

$$|<\Lambda_{k-1}^{k}>^{-1}|\le|<\Lambda_{k-1}^{k-1}>^{-1}| + 2d(\nu+1)\tau^{2}\,|H^{k}-H^{k-1}|_{C^{3},B_{2r}(K_{0})\times Q} \tag{17}$$

And from the theorem 1 in section 2, we get

$$|<\Lambda_{k-1}^{k-1}>^{-1}|<|<\Lambda_{k-2}^{k-1}>^{-1}|+\gamma^{-2}\delta_{k-1}{}^{2\sigma-1}2^{-4\sigma} \tag{18}$$

Appling (17),(18) repeatedly, we obtain

$$\begin{aligned}&|<\Lambda_{k-1}^{k}>^{-1}|\le\\ &\le|<\Lambda_{k-2}^{k-1}>^{-1}|+2d(\nu+1)\tau^{2}\,|H^{k}-H^{k-1}|_{C^{3},B_{2r}(K_{0})\times Q}\\ &+\gamma^{-2}\delta_{k-1}{}^{2\sigma-1}2^{-4\sigma}\le\cdots\le|<\Lambda_{0}>^{-1}|\\ &+\sum_{j=1}^{k}2d(\nu+1)\tau^{2}\,|H^{j}-H^{j-1}|_{C^{3},B_{2r}(K_{0})\times Q}\\ &+\sum_{j=1}^{k}\gamma^{-2}\delta_{j-1}{}^{2\sigma-1}2^{-4\sigma}\le\\ &\le|<\Lambda_{0}>^{-1}|+\sum_{j=1}^{\infty}2d(\nu+1)\tau^{2}\,|H^{j}-H^{j-1}|_{C^{3},B_{2r}(K_{0})\times Q}\\ &+\sum_{j=1}^{\infty}\gamma^{-2}\delta_{j-1}{}^{2\sigma-1}2^{-4\sigma}\le|<\Lambda_{0}>^{-1}|+1+\beta=\tau.\end{aligned}$$

(here, we used (10)). Therefore we obtain

$$|<\Lambda_{k-1}^{k-1}>^{-1}|\le|<\Lambda_{0}>^{-1}|+1+\beta=\tau.$$

Hence for any $k\in \boldsymbol{N}$, $\tau_{k}=|<\Lambda_{k-1}^{k}>^{-1}|\le\tau$ holds. From definition of c, we get $c_{k}\le c$. This implies A3(k).

Now let's prove A4(k). We get

$$|H^{k}-H^{k-1}|_{C^{3},B_{2r}(K_{0})\times Q}\le\|e_{0}\|_{\rho_{0}}(4^{l+2\sigma})^{-k+1}$$

from (16). Performing some computation, we obtain that

$$\begin{aligned}&c_{k}\gamma^{-4}\delta_{k}{}^{-4\sigma}\,\|e_{k}\|_{\rho_{k}}=c\gamma^{-4}(\delta_{0}/2^{k-1})^{-4\sigma}\\ &\|J\nabla H_{\lambda_{k-1}}^{k}(K_{k-1}(\theta))-\partial_{\omega}K_{k-1}(\theta)\|_{\rho_{k}}\le c\gamma^{-4}(\delta_{0}/2^{k-1})^{-4\sigma}\\ &(\|J\nabla H_{\lambda_{k-1}}^{k}(K_{k-1}(\theta))-J\nabla H_{\lambda_{k-1}}^{k-1}(K_{k-1}(\theta))\|_{\rho_{k}}+\\ &+\|J\nabla H_{\lambda_{k-1}}^{k-1}(K_{k-1}(\theta))-\partial_{\omega}K_{k-1}(\theta)\|_{\rho_{k}})=c\gamma^{-4}\delta_{0}{}^{-4\sigma}(2^{k-1})^{4\sigma}\\ &\|J\nabla H_{\lambda_{k-1}}^{k}(K_{k-1}(\theta))-J\nabla H_{\lambda_{k-1}}^{k-1}(K_{k-1}(\theta))\|_{\rho_{k}}\le\\ &\le c\gamma^{-4}\delta_{0}{}^{-4\sigma}\,|H^{k}-H^{k-1}|_{C^{3},B_{2r}(K_{0})\times Q}(2^{k-1})^{4\sigma}\le\\ &\le c\gamma^{-4}\delta_{0}{}^{-4\sigma}\,\|e_{0}\|_{\rho_{0}}(4^{l+2\sigma})^{-k+1}(2^{k-1})^{4\sigma}\le\\ &\le(4^{l+2\sigma})^{-k+1}(4^{k-1})^{2\sigma}\le(\frac{1}{4^{l}})^{k-1}\le 1.\end{aligned}$$

Therefore A4(k) holds. Similarly,

$$\begin{aligned}&c_{k}\gamma^{-2}\delta_{k}{}^{-2\sigma}\,\|e_{k}\|_{\rho_{k}}=c\gamma^{-2}(\delta_{0}/2^{k-1})^{-2\sigma}\\ &\|J\nabla H_{\lambda_{k-1}}^{k}(K_{k-1}(\theta))-\partial_{\omega}K_{k-1}(\theta)\|_{\rho_{k}}\le\\ &\le c\gamma^{-2}(\delta_{0}/2^{k-1})^{-2\sigma}\\ &(\|J\nabla H_{\lambda_{k-1}}^{k}(K_{k-1}(\theta))-J\nabla H_{\lambda_{k-1}}^{k-1}(K_{k-1}(\theta))\|_{\rho_{k}}+\\ &+\|J\nabla H_{\lambda_{k-1}}^{k-1}(K_{k-1}(\theta))-\partial_{\omega}K_{k-1}(\theta)\|_{\rho_{k}})=\\ &=c\gamma^{-2}\delta_{0}{}^{-2\sigma}(2^{k-1})^{2\sigma}\\ &\|J\nabla H_{\lambda_{k-1}}^{k}(K_{k-1}(\theta))-J\nabla H_{\lambda_{k-1}}^{k-1}(K_{k-1}(\theta))\|_{\rho_{k}}\le\\ &\le c\gamma^{-2}\delta_{0}{}^{-2\sigma}\,|H^{k}-H^{k-1}|_{C^{3},B_{2r}(K_{0})\times Q}(2^{k-1})^{2\sigma}\le\\ &\le c\gamma^{-2}\delta_{0}{}^{-2\sigma}\,\|e_{0}\|_{\rho_{0}}(4^{l+2\sigma})^{-k+1}(2^{k-1})^{2\sigma}\le\\ &\le r(4^{l+2\sigma})^{-k+1}(4^{k-1})^{\sigma}\le r(\frac{1}{4^{l+\sigma}})^{k-1}=r_{k}\end{aligned}$$

and this implies A5(k). Hence A1(k)-A5(k) hold for all $k\in\boldsymbol{N}$.

Let's λ_{k} and K_{k} are the parameter and the map obtained by applying the theorem in section 2 k-times. Then λ_{k} and K_{k} satisfies the following inequality from theorem 1 in section 2:

$$\|K_{k}-K_{k-1}\|_{\rho/4^{k}}\le\|K_{k}-K_{k-1}\|_{\rho_{k}}\le r_{k}=r\left(\frac{1}{4^{l+\sigma}}\right)^{k-1},$$

$$|\lambda_{k}-\lambda_{k-1}|<r_{k}=r(4^{-l+\sigma})^{k-1}.$$

Therefore $\{\lambda_{k}\}$ converges to a certain parameter $\lambda_{\infty}\in Q$. And from lemma 4, the sequence of real analytic maps $\{K_{k}\}$ converges to certain map $K_{\infty}\in C^{1}(\boldsymbol{T}^{n},\boldsymbol{U})$. Because for any $k\in\boldsymbol{N}$, K_{k} satisfy (1) with Hamiltonian $H_{\lambda_{k}}^{k}$ and H_{k} C^{3}-converges to H in $\boldsymbol{T}^{n}$, therefore K_{∞} is a solution of (1) with $H_{\lambda_{\infty}}$. □

4. CONCLUSION AND FURTHER STUDY

In this paper we generalized the result of [17] to the case of differentiable Hamiltonian vector fields with parameters.

In further study, we will try to weaken the limit on differentiability of Hamiltonian function in theorem 1.

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