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Eigensolutions Of The SchrÖdinger Equation And The Onicescu Information Energy With Similar Potentials

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Abstract

An approximate analytical solution of the non-relativistic Schrödinger equation for any arbitrary $\ell -$ states is studied with a combination of inverse potential and inversely quadratic Yukawa potential using the supersymmetric approach as the powerful tool to obtain the energy eigenvalue and the corresponding wave function in the presence of a suitable approximation scheme. The Onicescu information energy is also calculated in a close and compact form. The effect of the potential strengths' on the Onicescu information energy is investigated in detail. It is observed that as the Onicescu information energy decreases, the quantum number n increases.

Key words: Eigensolutions, Schrödinger equation; supersymmetric approach.

Introduction

In the recent years, the analytical approaches for solving the second order homogeneous linear differential equations have been proposed [1-7]. This is because they are less time consuming compared to the numerical techniques. These methods are powerful tools in finding the energy eigenvalues and wave functions (eigensolutions) of all solvable quantum physical potentials [8-14]. The different analytical techniques for finding the eigensolutions includes asymptotic iteration method (AIM), Nikiforov-Uvarov (NU) method, supersymmetric quantum mechanics approach (SUSY QM), shifted 1/N expansion method, factorization method, exact/proper quantization rule, formula method for bound state problem and so on. These methods have been used by various authors to solve Schrödinger equation, Dirac equation and Klein-Gordon equation with some potential models such as Manning-Rosen potential [4, 7, 15], Hyperbolic potential [16, 17], Pöschl-Teller potential [18-20], Deng-Fan potential [21-22], Tietz potential [23], Frost-Musulin potential [24], Yukawa potential [25], inversely quadratic Yukawa potential [26-27]. It is noted that there is little or no report on the Schrödinger equation with either inverse potential or inversely quadratic Yukawa potential. Thus, in this paper, we intend to study the Schrödinger equation with a combination of the inverse potential and inversely quadratic Yukawa potential via supersymmetric approach. The combination of these potential takes the form $V(r)=\frac{V}{r^2}-\frac{V_0 e^{-2\alpha r}}{r^2}$,

(1)

where V and V_0 are the potential depth and α is the range of the potential. It is obvious that the exact solution of the Schrödinger equation with potential (1) is not possible due to the centrifugal barrier. Thus, we resort to the use of approximation scheme given as

$$\frac{\ell(\ell+1)}{r^2}=\frac{4\ell(\ell+1)\alpha^2 e^{-2\alpha r}}{\left(1-e^{-2\alpha r}\right)^2}. \tag{2}$$

2. Bound State Solution

Given a three dimensional Schrődinger equation [28] as

$$[-\frac{\hbar^2}{2\mu}(\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}+\frac{1}{r^2 sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2 sin^2\theta}\frac{\partial^2}{\partial\varnothing^2})+V(r)]\psi_{n\ell}(r)=E\psi_{n\ell}(r) \tag{3}$$

and setting the wave function $\psi_{n\ell m}(r)=U_{n\ell}(r)Y_{m\ell}(\theta,\phi)r^{-1}$, we obtain the radial part of the Schrődinger equation by the separation of variables as

$$\left[\frac{d^2}{dr^2}+\frac{2\mu}{\hbar^2}\left(E_{n,\ell}-V_{eff}(r)\right)\right]U_{n,\ell}(r)=0, \tag{4}$$

$$V_{eff}(r)=V(r)+\frac{\ell(\ell+1)\hbar^2}{2\mu r^2},$$

and then substituting potential (1) and approximation (2) into Eq. (4), we have

$$\frac{d^2U_{n,l}(r)}{dr^2}=\left\{\frac{2\mu}{\hbar^2}\left(-E_{n,l}+V_1\right)+V_2\right\}U_{n,l}(r). \tag{5}$$

$$V_1=\frac{4V_0\alpha^2 e^{-2\alpha r}}{\left(1-e^{-2\alpha r}\right)}+\frac{4\left(V-V_0\right)\alpha^2 e^{-2\alpha r}}{\left(1-e^{-2\alpha r}\right)^2},$$

$$V_2=\frac{4\ell(\ell+1)\alpha^2 e^{-2\alpha r}}{\left(1-e^{-2\alpha r}\right)^2}.$$

To obtain the solution of Eq. (5), we proposed a super potential function of the form

$$Q(r)=\rho_1+\frac{\rho_2 e^{-2\alpha r}}{1-e^{-2\alpha r}}, \tag{6}$$

By applying the basic concepts of the supersymmetric quantum mechanics formalism and shape invariance technique [29-34] to solve Eq. (5), the ground state function can be written in the form

$$U_{0,\ell}(r)=exp\left(-\int Q(r)dr\right), \tag{7}$$

where a relation

$$Q^2(r)-\frac{dQ(r)}{dr}=\frac{d^2U_{n,\ell}(r)}{dr^2}., \tag{8}$$

is defined. Substituting Eq. (6) into Eq. (8), we obtain the values of the two parametric constants in Eq. (6) as follow:

$$\rho_1=\frac{\frac{8\mu V_0\alpha^2}{\hbar^2}}{2\rho_2}-\frac{\rho_2}{2}, \tag{9}$$

$$\rho_2=-\alpha\left[1\pm\sqrt{(1+2\ell)^2+\frac{8\mu\left(V-V_0\right)}{\hbar^2}}\right]. \tag{10}$$

Eq. (8) is a non-linear Riccati equation whose solution is obtained by inserting a superpotential function of Eq. (6). The superpotntial function results to the formation of partner potentials. Thus, to proceed to the next level using the shape invariance formalism, it is very important to construct the partner potentials using the superpotential function:

$$V_+(r)=Q^2(r)+\frac{dQ}{dr}=\rho_1^2+$$
$$\frac{\rho_2(2\rho_1-\rho_2)e^{-2\alpha r}}{1-e^{-2\alpha r}}+\frac{\rho_2(\rho_2+2\alpha)e^{-2\alpha r}}{\left(1-e^{-2\alpha r}\right)^2},$$

(11)

$$V_+(r)=Q^2(r)+\frac{dQ}{dr}=\rho_1^2+$$
$$\frac{\rho_2(2\rho_1-\rho_2)e^{-2\alpha r}}{1-e^{-2\alpha r}}+\frac{\rho_2(\rho_2-2\alpha)e^{-2\alpha r}}{\left(1-e^{-2\alpha r}\right)^2}. \quad (12)$$

Putting $\rho_2=a_0$, then, the partner Hamiltonian are shape invariant via mapping of the form $\rho_2\to\rho_2-2\alpha$. If the shape invariance condition holds [31], then,

$$V_+(a_0,r)=V_-(a_1,r)+R(a_1), \quad (13)$$

where a_1 is a new set of parameters uniquely determined from an old set a_0 via the mapping F: $a_0\to a_1=f(a_0)$ and the residual term $R(a_1)$ is independent of the variable r. Therefore, $a_1=a_0-2\alpha$ and subsequently, $a_n=a_0-2\alpha n$. The energy eigenvalues of the Hamiltonian

$$H_+(r)=-\frac{d^2}{dr^2}+V_+(r), \quad (14)$$

are given by

$$\frac{-2\mu E_{n,l}}{\alpha^2\hbar^2}=R(a_1)+R(a_2)+R(a_3)+,$$
$$-,-,-,R(a_{n-1})+R(a_n). \quad (15)$$

Then, from Eqs. (9) and (10), we obtain the relation:

$$\frac{-2\mu E_{n,l}}{\alpha^2\hbar^2}=\left(\frac{\frac{8\mu V_0\alpha^2}{\hbar^2}}{2\rho_2}-\frac{\rho_2}{2}\right)^2. \quad (16)$$

Substituting the value of ρ_2 into Eq. (16) by taking $\rho_2=\rho_2-2\alpha n$ we obtain the complete energy spectrum as

$$E_{n\ell}=-\frac{\alpha^2\hbar^2}{2\mu}\left[\frac{\frac{2\mu V_0}{\hbar^2}+\aleph^2}{\aleph}\right]^2. \quad (17)$$

$$\aleph=n+\frac{1}{2}+\frac{1}{2}\sqrt{(1+2\ell)^2+\frac{8\mu(V-V_0)}{\hbar^2}}$$

The corresponding wave function is

$$U_{n\ell}(r)=N_{n\ell}y^{\lambda}(1-y)^{\eta}\times$$
$${}_2F_1\left(-n,n+2(\lambda+\eta)+2\left(\lambda+\frac{1}{2}\right),y\right), \quad (18)$$

where, $y=e^{-2\alpha r}$,

$\lambda=\sqrt{-\frac{\mu E_{n\ell}}{2\alpha^2\hbar^2}-\frac{2\mu V_0}{\hbar^2}}$ and

$$\eta=\frac{1}{2}+\frac{1}{2}\sqrt{(1+2\ell)^2+\frac{8\mu(V-V_0)}{\hbar^2}}.$$

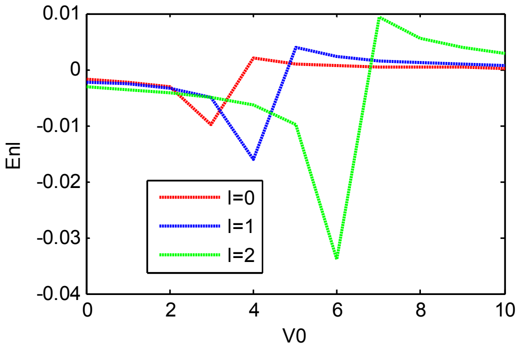

Fig. 1: Variation of energy with the inversely quadratic Yukawa potential strength.

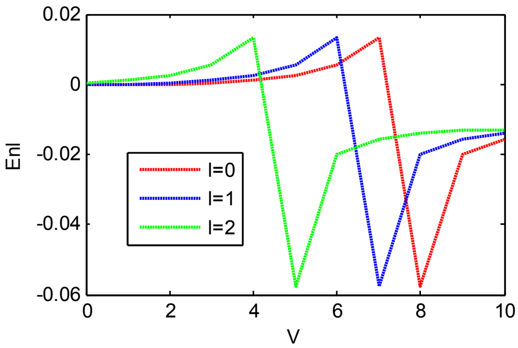

Fig. 2: Variation of energy with the inverse potential strength.

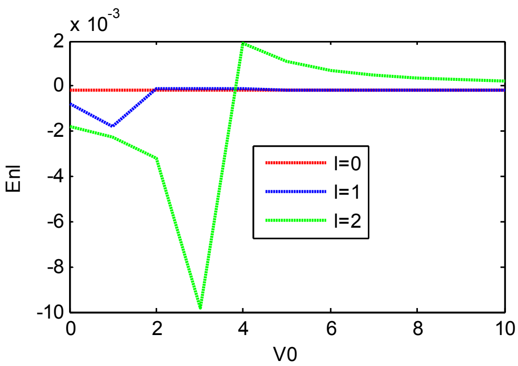

Fig. 3: Variation of the energy of Yukawa potential with its potential strength.

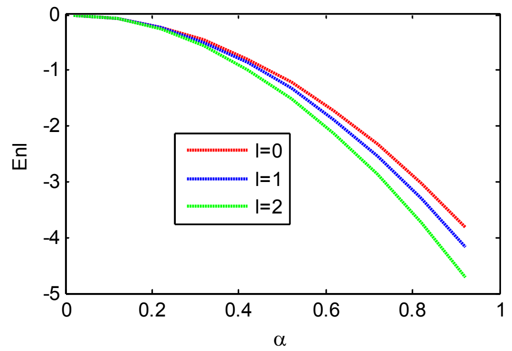

Fig. 4: Energy against potential range.

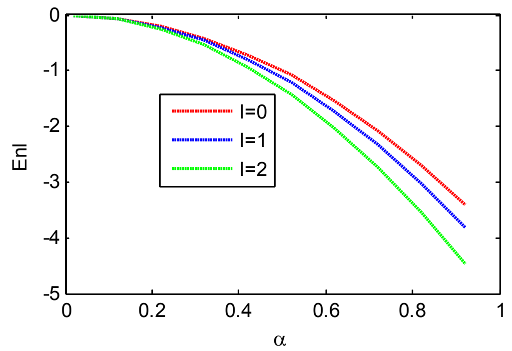

Fig. 5: Energy of the inversely quadratic Yukawa potential against its potential range.

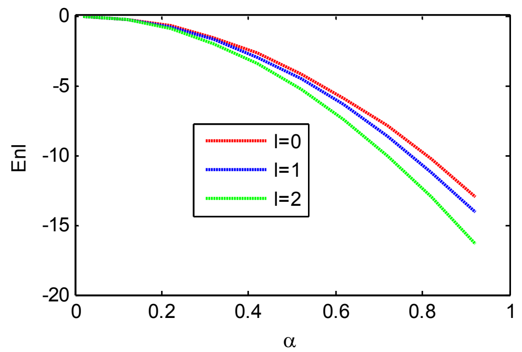

Fig. 6: Energy against potential range for inverse potential.

3. Onicescu information energy

The Onicescu information energy is defined as [35]

$$E(\rho) = 4\pi\int_0^{\infty} \rho^2(r). \tag{19}$$

$$E(\rho) = -\frac{2\pi}{\alpha}\int_1^0 \rho^2(y)dy,\ y = e^{-2\alpha r}, \tag{20}$$

$$E(\rho) = \frac{\pi}{\alpha}\int_{-1}^1 \rho^2(s)ds,\ z = 1-2y, \tag{21}$$

Now, defining a relation of the form

$$\square\ {}_2F_1\left(-n, n+2(\lambda+\zeta)+2\left(\lambda+\frac{1}{2}\right), e^{-\delta r\lambda}\right) = \square \times \left[P_n^{(\lambda,\zeta)}(x)\right]^2$$

(22)

then, the probability distribution which is equal to the squared or normalized radial wave function is given as

$$\rho^2(z) = \left(\frac{1-z}{2}\right)^{\frac{1}{2}\lambda}\left(\frac{1+z}{2}\right)^{2\eta-1} \times \left[P_n^{(\lambda,\eta)}(z)\right]^2. \tag{23}$$

where,

$$\square = e^{-\delta r\lambda}\left(1-e^{-\delta r}\right)^{\zeta}$$

Using standard integral of the form

$$\int_{-1}^1 \left(\frac{1-x}{2}\right)^{a-1}\left(\frac{1+x}{2}\right)^{b} \times \left[P_n^{(a,b)}(x)\right]^2 dx = \frac{2\Gamma(a+n+1)\Gamma(b+n+1)}{n!\,a\Gamma(a+b+n+1)}, \tag{24}$$

$$E(\rho) = \frac{3.142}{\alpha} \times \left(\frac{1}{\Gamma(0.5\lambda+2\eta+n)}\right)^2, \tag{25}$$

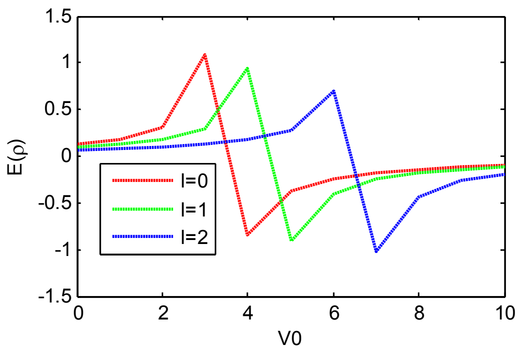

Fig. 7: Information energy against the strength of inversely quadratic Yukawa potential with $V = 3$ and $\alpha = 0.2.$

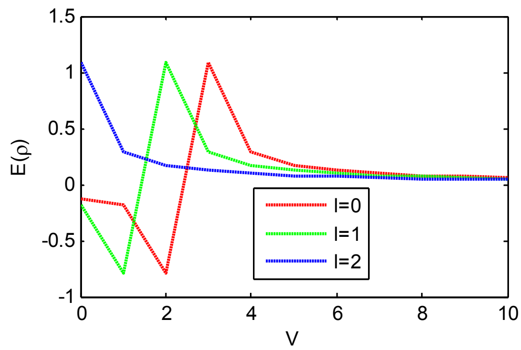

Fig. 8: Information energy against the strength of inverse potential with $V_0 = 3$ and $\alpha = 0.2.$

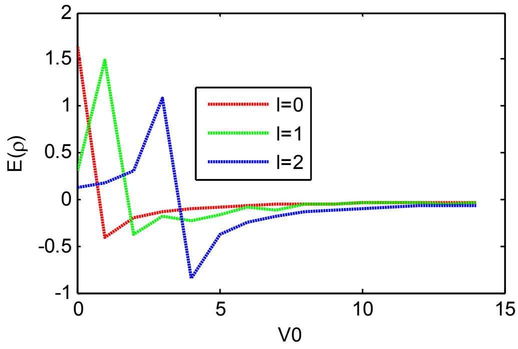

Fig. 9: Information energy of inversely quadratic Yukawa potential against its strength with $\alpha = 0.2.$

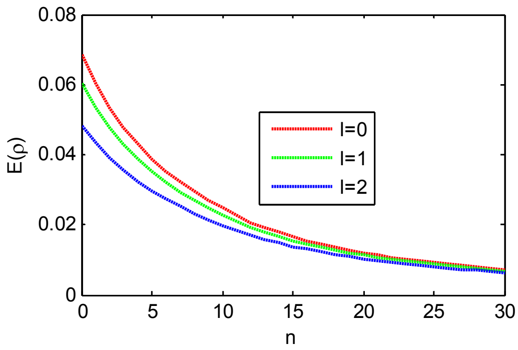

Fig. 10: Information energy against the quantum number n with $V = 8,\ V_0 = 1$ and $\alpha = 0.2.$

$$N_{nk}^2 = \frac{n! \times 0.5\lambda \times \Gamma(0.5\lambda + 2\eta + n)}{2\Gamma(0.5\lambda + n + 1)\Gamma(2\eta + n)}. \qquad (26)$$

4. Discussion

The solutions of other useful potentials like the inversely quadratic Yukawa potential and the inverse potential are obtained by changing the numerical values of the two potential strength. When we put $.V_0 = 0$, the potential (1) reduces to the inverse potential and the energy equation 17 becomes

$$E_{n\ell} = -\frac{\alpha^2\hbar^2}{2\mu}\left[\frac{\left(n + \frac{1}{2} + \frac{1}{2}\Re\right)^2}{n + \frac{1}{2} + \frac{1}{2}\Re}\right]^2. \qquad (27)$$

where

$$\Re = \sqrt{(1 + 2\ell)^2 + \frac{8\mu V}{\hbar^2}}$$

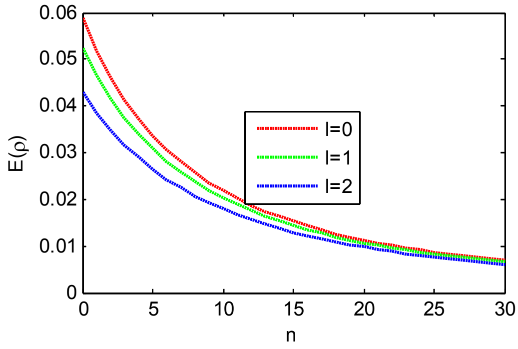

Fig. 11: Information energy of the Inverse potential against the quantum number n with $\alpha = 0.2.$

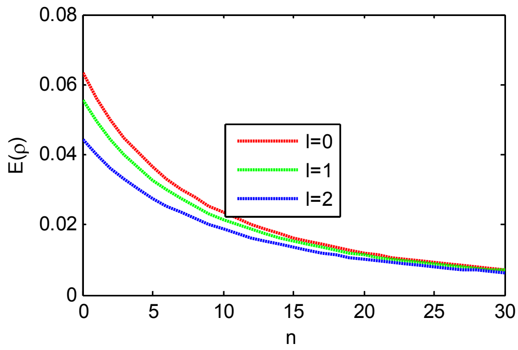

Fig. 12: Information energy of the Yukawa potential against the quantum number n with $\alpha = 0.2.$

Similarly, when $.V = 0,$ our potential (1) reduces to the inversely quadratic Yukawa potential and the energy equation (17) turns to be

$$E_{n\ell} = -\frac{\alpha^2\hbar^2}{2\mu}\left[\frac{\frac{2\mu V_0}{\hbar^2} + \left(n + \frac{1}{2} + \frac{1}{2}\Re\right)^2}{n + \frac{1}{2} + \frac{1}{2}\Re}\right]^2. \qquad (28)$$

where

$$\Re = \sqrt{(1 + 2\ell)^2 - \frac{8\mu V_0}{\hbar^2}}.$$

In Figs 1 and 2, we plotted the energy eigenvalue of potential (1) against the potential depths V_0 and V respectively. In Fig 3, the variation of the energy of the inversely quadratic Yukawa potential with its depth is shown. In Fig. 4-6, we plotted the energy of potential (1), inversely quadratic Yukawa potential and the inverse potential respectively with the potential range. In both cases, similar characteristics are observed. In Figs 7 and 8, we plotted Onicescu information energy of the potential

(1) with the two potential depth. In Fig. 9, the information energy of the inversely quadratic Yukawa potential against the potential depth of Yukawa potential is plotted. It is observed that the variation of the energy eigenvalues with the potential depth is similar to the variation of the Onicescu information energy with the potential depth. In Fig. 10 -12, we plotted Onicescu information energy with the quantum number n for the potential (1), inversely quadratic Yukawa potential and inverse potential respectively. In both cases, the Onicescu information decreases as the quantum number increases.

5. Conclusion

In this paper, we have studied the bound state solutions of the Schrödinger equation with an interaction of inverse and inversely quadratic Yukawa potentials via the supersymmetric method. It is observed that the effects of the strength of the inversely quadratic Yukawa potential on energy eigenvalues is similar to the effects of the strength of inverse potential on the Onicescu information energy while the effects of the strength of inverse potential on the energy eigenvalues is similar to the effects of the strength of Yukawa potential on the Onicescu information energy. The Onicescu information energy decreases as the quantum number n increases.

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